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CONDITIONED LIMIT THEOREMS FOR SOME NULL RECURRENT MARKOV PROCESSES

BY

RICHARD DURRETT

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Chapter 1

INTRODUCTION

1.1 Summary of Results

Let $\{v_k, k \geq 0\}$ be a discrete time Markov process with state space $E \subset (-\infty,\infty)$ and let S be a proper subset of E. In several applications (see [8], [12], and [13]) it is of interest to know the behavior of the system after a large number of steps given the process has not entered S. For example, if v_n is a branching process and $S = \{0\}$ a limit theorem for $(v_n | v_m \neq 0 \quad 1 \leq m \leq n)$ gives information about the size of v_n on the set $\{v_n > 0\}$.

In [2], Seneta and Vere-Jones have given conditions for the convergence of

$$\alpha_{ij}(n) \quad = \quad P(v_n = j | v_0 = i, N_S > n) \tag{1}$$

where $N_S = \inf\{m \geq 1 : v_m \in S\}$. In many cases, however, all the limits in (1) are zero. Applying the results of [2] when v_n is a branching process and $S = \{0\}$ gives that $\alpha_j^* = \lim_{n \to \infty} \alpha_{ij}(n)$ is a probability distribution when $m = E(v_1 | v_0 = 1) < 1$ and $\alpha_j^* \equiv 0$ when $m \geq 1$. To obtain an interesting theorem in the second case we have to look at the limit of $(v_n/c_n | v_0 = i, N_S > n)$ where the c_n are constants which $\uparrow \infty$.

In this instance the most desirable type of result is a functional limit theorem, i.e., a result asserting the convergence of the sequence of stochastic processes $\{V_n^+(t), 0 \leq t \leq 1\}$ defined by

$$V_n^+(t) \quad = \quad (v_{[nt]}/c_n | v_0 = i, N_S > n) \tag{2}$$

where $[x]$ is the largest integer $\leq x$. This was the goal in the applications cited above but in each case the results given are incomplete due to problems with the tightness argument.

It was the presence of these technical difficulties which motivated this investigation. The techniques we have developed allow us to complete the work mentioned above. While writing out the solutions to these problems we noticed that the arguments we were giving had many aspects in common. To determine which properties were used and how they contribute to the proof, we isolated the hypotheses as numbered assumptions and studied their relationships and consequences. As a result of this we were able to formulate general conditions for the process V_n^+ to converge when $S = (-\infty, 0] \cap E$.

There are two advantages of deriving our conclusions from a set of basic assumptions. The first is obvious: a person who is interested in proving a conditional limit theorem may apply our results directly instead of having to modify our proofs to meet his needs. A second, less tangible, benefit is that the arguments we give do not depend upon special properties of the Markov chain and so the proofs may contribute to an intuitive understanding of the conditions needed to guarantee convergence.

It is the second idea which has been our guide in the developments below. Our aim has been to find assumptions which create a sharp division into cases, i.e., so that the limit theorems hold under the assumptions given and fails or is trivial in the other cases. To describe the extent of our success we have to explain our results in some detail.

We begin by stating our three basic assumptions: (i) v_k, $k \geq 0$ is a Markov process with state space $E \subset (-\infty,\infty)$; (ii) there are constants $c_n \uparrow \infty$ with $c_{n+1}/c_n \to 1$ so that if $x_n \to x$ and $x_n c_n \in E$ for all n then

$$V_n^{x_n} = (v_{[n\cdot]}/c_n \mid v_0/c_n = x_n) \Rightarrow (V \mid V(0) = x) = V^x$$

where V is a Markov process with V^y nondegenerate for some $y > 0$ and (iii) $P\{\inf_{0 \leq s \leq t} V^x(s) > 0\} > 0$ for all $t, x > 0$.

Here the symbol $\Rightarrow$ means that the sequence $V_n^{x_n}$ converges weakly as a sequence of random elements of D - the space of right continuous functions on $[0,1]$ which have left limits.* Nondegenerate means that $P\{V^x = f\} < 1$ for all $f \in D$.

Let $N = N_{(-\infty,0]}$. It is under assumptions (i)-(iii) that we will derive conditions for the convergence of $(V_n^{x_n} \mid N > n)$ (a) for all $x_n \to x \geq 0$ and (b) when $x_n c_n \equiv y \in E$.

We will obtain our conditions for the case $x_n \to x > 0$ by solving a more general problem. In Section 2 we give sufficient conditions for the convergence of $P_n(\cdot \mid A_n) = P_n(\cdot \cap A_n)/P_n(A_n)$ when the P_n are probability measures with $\inf_n P_n(A_n) > 0$. Applying these results to sets $A_n = \{f: \inf_{0 \leq s \leq t_n} f(s) > 0\}$ with $t_n \to t \in [0,1]$ we find that if $P_n^{x_n}$ and P^x are the probability measures induced on D by $V_n^{x_n}$ and V^x, and $x_n \to x > 0$ then $P_n^{x_n}\{N > nt_n\} \to P^x\{T_0 > t\}$ is sufficient for $(V_n^{x_n} \mid N > nt_n) \Rightarrow (V^x \mid T_0 > t)$ when $T_0 = \inf\{t > 0: \inf_{t/2 \leq s \leq t} f(s) \leq 0\}$.

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*In Section 2.1 there is a brief description of this space and the weak convergence results used in this paper. Most of the results we will need can be found in [20].

If $x_n \to 0$, however, $P_n^{x_n}\{N > n\} \to 0$ (in most cases) so a more delicate analysis is required. Our method for proving convergence in this case will be to show that if $T_\varepsilon^n = \inf\{k : v_k/c_n \geq \varepsilon\}$ then

$$\lim_{n\to\infty} (v_{[n\cdot]}/c_n \mid v_0 = x_n c_n,\ N > n)$$

$$= \lim_{\varepsilon\to 0}\ \lim_{n\to\infty} (v_{[T_\varepsilon^n + n\cdot]}/c_n \mid v_0 = x_n c_n,\ N > n)$$

$$= \lim_{\varepsilon\to 0}\ \lim_{n\to\infty} (v_{[n\cdot]}/c_n \mid v_0 = \varepsilon c_n,\ N > n)$$

$$= \lim_{\varepsilon\to 0} (V^{\varepsilon} \mid T_0 > 1)$$

In Section 3 we will show that these three equalities hold if (in addition to (i)-(iii)) we have

(iv) $P_n^{x_n}\{N > nt_n\} \to P^x\{T_0 > t\}$ whenever $x_n \to x > 0$, $t_n \to t > 0$ and

(v) $P_n^{x_n}\{N > nt_n\} \to 0$ whenever $x_n \to 0$ and $t_n \to t > 0$.

The key to our proof is the following fact (first observed by Lamperti in [25]):

<u>Theorem 3.2</u> If (i) and (ii) hold there is a $\delta \geq 0$ so that for all $c > 0$ $V^{cx} \overset{d}{=} cV^x(\cdot c^\delta)$. (*)

This sealing relationship identifies the processes which can occur as limits in (ii) and can be used to deduce many properties of the limit process. In Section 3.1 we use (*) to compute relationships between the numbers $P^x\{T_0 > t\}$. These formulas are used to identify trivial cases and obtain sufficient conditions for (iii), (iv), and (v) to hold.

In Sections 3.2 and 3.3 we use these preliminaries to prove our conditional limit theorems. To do this we reverse the usual procedure for proving weak convergence. In Section 3.2 we develop sufficient conditions for V_n^+ to be tight. In Section 3.3 we find conditions for the convergence of finite dimensional distributions:

<u>Theorem 3.10</u> Suppose (i)-(iv) hold and V_n^+ is tight. If $V^+ = \lim_{x \downarrow 0} (V^x | T_0 > 1)$ exists as is $\not\equiv 0$ then $V_n^+ \Rightarrow V^+$ if and only if

$$\lim_{\delta \downarrow 0} \varliminf_{n \to \infty} P\{V_n^+(t) > \delta\} = 1 \quad \text{for all} \quad t > 0$$

If $x_n c_n \equiv a$ and (v) holds this condition is equivalent to

$$\lim_{n \to \infty} \frac{P\{N > nt | v_0 = a\}}{P\{N > n | v_0 = a\}} = t^{-\beta} \quad \text{for some} \quad \beta \geq 0 \text{ .}$$

In Sections 4.1-4.4 we use the results of Section 3 to prove conditioned limit theorems for random walks, branching processes, birth and death processes, and the M/G/1 queue which contain the corresponding results of [6], [8], [12] and [13] as special cases. It seems likely that our methods can be extended for the non-Markovian examples studied by [7] and [11], but we have not tried this.

A more interesting unsolved problem is to prove that if $S_n = \sum_{i=1}^{n} X_i$ is a random walk with $EX_1 = u < 0$, $E(X_1 - u)^2 = \sigma^2 < \infty$ and $P\{X_1 > 0\} > 0$ then $(S_{[n\cdot]}/\sigma n^{1/2} | S_0 = 0, N_{(-\infty,0]} > n)$ converges to the Brownian bridge (see [20] p. 64 for a description). Conditions for convergence are known if $E(e^{\theta x_1}) < \infty$ for $\theta \epsilon (-a,a)$ (see [4]) but methods given here cannot be applied since (ii) does not hold for $c_n = \sigma n^{1/2}$.

In Section 4.5 we show that the developments in Section 3 can be modified to prove the results of [5] and [14] for random walks conditioned on $\{N_B > n\}$ when B is a bounded subset of the state space. This example suggests that our results may be extended to conditioning to avoid other types of sets S. Unfortunately there are no other possibilities. It is easy to show that if $(v_n/c_n | v_0=y,\ N_S > n)$ converges then $\bigcap_{m=1}^{\infty} (\bigcup_{n=m}^{\infty} Sc_n^{-1})^-$ is $\{0\}$, $(-\infty,0]$, $[0,\infty)$, or $(-\infty,\infty)$ so we have already considered the two reasonable cases.

To generalize our results we can consider other types of conditioning. A natural candidate for this is conditioning on $\{v_n \in A\}$ or $\{(v_{n-1}, v_n) \in B\}$. Several limit theorems of this type have appeared in the literature with $A = \{x\}$ or $[a,b]$ (see [15]-[18]) and $B = (-\infty,0) \times (0,\infty)$ (see [19]) and it seems that our methods can be applied. These conditionings have the most effect at times close to 1, however, so we have to reverse our perspective and new techniques are required. We plan to consider these limit theorems in a later publication

1.2 Weak Convergence and the Geometry of D

Let (S,ρ) be a metric space and $\mathcal{S}$ the class of Borel subsets of S. If P_n, $n \geq 0$ are probability measures on $\mathcal{S}$ and $\int f dP_n \to \int f dP_0$ for every bounded continuous f on S then we say P_n converges weakly to P_0 and write $P_n \Rightarrow P_0$. There are, of course, many other definitions of weak convergence.

Theorem 1. The following four conditions are equivalent:

(i) $P_n \Rightarrow P$

(ii) $\overline{\lim_n} \int f dP_n \leq \int f dP$ for all bounded upper semicontinuous f

(iii) $\underline{\lim_n} P_n(G) \geq P(G)$ for all open sets G and

(iv) $P_n(A) \to P(A)$ for all A with $P(\partial A) = 0$.

This and most of the other weak convergence results we will need can be found in [20] or are given in Section 2. In addition to the standard results, however, we will need some special facts about the geometry of D which are not available in an easily quotable form. These results and some related well-known convergence notions are explained below. Proofs are given only for results which cannot be found in [20].

Let D be the space of functions on $[0,1]$ that are right continuous and have left limits. Let Λ denote the class of strictly increasing continuous mapping from $[0,1]$ onto itself. For f and g in D define $d(f,g)$ to be the infimum of those positive ε for which there exists a $\lambda \in \Lambda$ such that

$$\sup_t |\lambda(t) - t| \le \varepsilon \tag{1}$$

and

$$\sup_t |f(t) - g(\lambda(t))| \le \varepsilon \tag{2}$$

It is easy to show that d is a metric for D ([20], p. 111). Many facts about the resulting topology for D are given in Chapter 14 of [20]. Two of these results which we will need later are:

$$f \to \sup_{0 \le t \le 1} f(t) \text{ is a continuous function} \tag{3}$$

$$\text{if } \pi_t(f) = f(t) \text{ then } \pi_t \text{ is continuous if}$$

$$\text{and only if } t = 0 \text{ or } t = 1 \tag{4}$$

For this study we will need information about the continuity of other functionals $h:D \to R$. The first we shall investigate is the modulus of continuity $\omega'_f(\delta) = \omega'_f(\delta;0,1)$ defined by

$$\omega'_f(\delta;a,b) = \inf_{\{t_i\}} \left[\max_{1 \le i \le r} \left(\sup_{t_{i-1} \le s < t < t_i} |f(s)-f(t)| \right) \right], \tag{5}$$

where the infimum is taken over all sequences

$$a \le t_0 \le t_1 \ldots < t_r = b \text{ with } \min_i (t_i - t_{i-1}) > \delta$$

<u>Theorem 2</u>. $f \to \omega'_f(\delta)$ is an upper semicontinuous function.

<u>Proof</u>.

Let $\eta > 0$. Suppose t_i are chosen for f so that the expression in (5) is less than $\omega'_f(\delta) + \eta$. If $d(f,g) < \varepsilon_0 = \eta \wedge (\min_{1 \le i \le r} t_i - t_{i-1} - \delta)/2$ and $\lambda \in \Lambda$ is such that (1) and (2) hold for $\varepsilon = \varepsilon_0$, using $\lambda(t_i)$ in (5) gives $\omega'_g(\delta) < \omega'_f(\delta) + 2\eta$.

For the proof of Theorem 3.3 we will need to know about the continuity of the hitting times which we define for $y > 0$ by

$$T_y(f) = \inf\{t > 0 : f(t) \geq y\}$$

It is easy to construct examples which show T_y is not lower or upper semicontinuous:

For $1 \leq n \leq \infty$ let

$$f_n(x) = \begin{cases} (3 + 1/n)x & 0 \leq x < 1/3 \\ x + 1/3 & 1/3 \leq x \leq 1 \end{cases}$$

$$g_n(x) = \begin{cases} (3 - 1/n)x & 0 \leq x \leq 1/3 \\ g_n(1/3) & 1/3 < x < 2/3 \\ x + 1/3 & 2/3 \leq x \leq 1 \end{cases}$$

All is not lost, however. The next result shows that almost every T_y is almost surely continuous.

<u>Theorem 3.</u> Let Δ_y be the set of discontinuities of T_y. If P is a probability measure on D then $\{y > 0 : P(\Delta_y) > 0\}$ is a set of Lebesque measure zero.

<u>Proof.</u>

Let
$$T_y^+(f) = \inf\{t > 0 : f(t) > y\}$$

$$T_y^\ell(f) = \inf\{t > 0 : \sup_{s \leq t} f(s) \geq y\}$$

Clearly, $T_y^\ell(f) \leq T_y(f) \leq T_y^+(f)$.

Lemma 1. T_y^+ is upper semicontinuous.

Proof.

If $T_y^+(f) < \infty$ then for any $\eta > 0$ there is a positive $s < T_y^+(f) + \eta$ so that $f(s) > y$. If $d(f,g) < \varepsilon_0 = (f(s)-y)\wedge\eta$ and $\lambda\in\Lambda$ is such that (1) and (2) hold for $\varepsilon = \varepsilon_0$ then $g(\lambda(s)) > y$ and $\lambda(s) > 0$ so $T_y^+(g) < T_y^+(f) + 2\eta$.

Lemma 2. T_y^ℓ is lower semicontinuous.

Proof.

If $T_y^\ell(f) = 0$ the conclusion is obvious. If $T_y^\ell(f) = \infty$ then $\sup_t f(t) < y$ so if $d(f,g) < y-\sup_t f(t)$, $T_y^\ell(g) = \infty$. If $0 < T_y^\ell(f) < \infty$ then for any positive $s < T_y^\ell(f)$, $\sup_{t\le s} f(t) < y$. If $\eta\in(0,s)$, $d(f,g) < \varepsilon_0 = (y-\sup_{t\le s} f(t))\wedge\eta$ and $\lambda\in\Lambda$ is such that (1) and (2) hold for $\varepsilon = \varepsilon_0$ then $\sup\{g(t):t \le \lambda(s)\} < y$ so $T_y^\ell(g) > \lambda(s) \ge s-\eta$.

Lemma 3. If P is a probability measure on D then $\{y > 0:P\{f:T_y^\ell(f) < T_y^+(f)\} > 0\}$ has Lebesque measure zero.

Proof.

Observe that $T_y^\ell(f) < T_y(f)$ only if $f(0) = y$ or f is discontinuous at $T_y^\ell(f)$, so for any f there are only a countable number of values for which strict inequality holds (see [20], p. 124).

For the other half observe that the intervals $[T_y(f),T_y^+(f))$ are disjoint for different y so only countably many are not empty.

Combining this with the first observation gives $\{y: T_y^-(f) < T_y^+(f)\}$ is countable for each f so applying Fubini's theorem gives the desired result.

The preceeding theorem is useful for proofs in which we have some choice in deciding which T_y to use. The examples above however show that we can in general conclude nothing about a specific hitting time of interest (say the time to hit $(-\infty,0]$). As a partial remedy we will define the hitting times in a slightly different manner for $y = 0$.

$$T_0^+(f) = \inf\{t > 0 : f(t) > 0\}$$

$$T_0(f) = \inf\{t > 0 : \inf_{t/2 \le s \le t} f(s) \le 0\}$$

$$T_0'(f) = \inf\{t > 0 : f(t) \le 0\}$$

$$T_0^-(f) = \inf\{t > 0 : f(t) < 0\}$$

We will work with T_0 instead of the "natural" hitting time T_0' since

$$\{f : f(0) > 0, T_0(f) > t\} = \{f : \inf_{0 \le s \le t} f(s) > 0\}$$

is open (a fact which is useful in Section 2). Observe that if $P^x, x \in (-\infty,\infty)$ are the transition probabilities of a standard Markov process (see [22], 9.2.v) then $P^x\{T_0 = T_0'\} = 1$ for all $x > 0$.

Chapter 2

CONDITIONS FOR THE CONVERGENCE OF $P_n(\cdot|A_n)$ WHEN $\inf_n P_n(A_n) > 0$

In this section we shall investigate conditions under which the weak convergence of a sequence of probability measures P_n is sufficient for the convergence of the conditional measures $P_n(\cdot|A_n) = P_n(\cdot \cap A_n)/P_n(A_n)$ when $\inf_n P_n(A_n) > 0$.

If $P_n(A_n) \to P(A)$, we can check that $P_n(\cdot|A_n) \Rightarrow P(\cdot|A)$ by showing that $P_n(B \cap A_n) \to P(B \cap A)$ for enough sets B. Sufficient conditions for this are an easy consequence of a generalization of the continuous mapping theorem ([20]Th.5.5).

To state this theorem requires some notation: let (S,ρ) and (S',ρ') be complete separable metric spaces with Borel fields $\mathcal{S}$ and $\mathcal{S}'$. Let h_n, $n \geq 0$ be measureable mappings from S to S' and let E be the set of $x \in S$ such that $h_n(x_n) \to h_0(x)$ fails to hold for some sequence $x_n \to x$.

<u>Theorem 1</u>. If $P_n \Rightarrow P_0$ with $P_0(E) = 0$ then $P_n h_n^{-1} \Rightarrow P_0 h_0^{-1}$. If the h_n are real valued and there is an $M < \infty$ so that $|h_n(x)| \leq M$ for all $n \geq 0$ and $x \in S$ then for any B with $P(\partial B) = 0$

$$\int_B h_n(x) P_n(dx) \to \int_B h_0(x) P_0(dx)$$

<u>Proof</u>.

This result can be easily obtained from the Skorohod representation theorem.

Lemma 1. If P_n, $n \geq 0$ are probability measures on the complete separable metric space S such that $P_n \Rightarrow P_0$ then there are Borel measurable $X_n:(0,1) \rightarrow S$ so that X_n has distribution P_n and X_n converges to X_0 almost surely (with respect to Lebesque measure).

If $P_0(E) = 0$ and X_n, $n \geq 0$ are the variables of the lemma for $P_n, n \geq 0$ then $h_n(X_n) \rightarrow h_0(X_0)$ almost surely so $P_n h_n^{-1} \Rightarrow P_0 h_0^{-1}$. To obtain the second result note that $1_{\{X_n \in B\}} h_n(X_n) \rightarrow 1_{\{X_0 \in B\}} h_0(X_0)$ almost surely and use the bounded convergence theorem.

To use this theorem to obtain a conditioned limit theorem let $h_n = 1_{A_n}$, the function which is 1 on A_n and 0 on A_n^c. If $P_0(E) = 0$, then Theorem 1 implies that $P_n(B \cap A_n) \rightarrow P_0(B \cap A_0)$ for all B with $P_0(\partial B) = 0$. To confirm that this is enough to guarantee $P_n(\cdot|A_n) \Rightarrow P_0(\cdot|A_0)$ we use the following lemma ([20], Cor 1, p. 14) with $U = \{B:P_0(\partial B) = 0\}$.

Lemma 2. A sequence of probability measures Q_n converges weakly to a limit Q if there is a class of sets U so that

(a) U is closed under finite intersections;

(b) for every $x \in S$ and $\varepsilon > 0$ there is a B is U with $x \in B^0$ (the interior of B) and $B \subset \{y:\rho(x,y) < \varepsilon\}$; and

(c) $Q_n(B) \rightarrow Q(B)$ for every B in U.

To translate $P_0(E) = 0$ into a condition on the sequence A_n note that $x \in E^c$ if and only if there is a k and $\delta > 0$ so that $\rho(x,y) < \delta$ and $n \geq k$ implies $h_0(x) = h_n(y)$. If $h_0(x) = 1$ then

$h_0(x) = h_n(y)$ for all $n \geq k$ means $y \in \bigcap_{n \geq k} A_n$ so in this case $x \in E^c$ if and only if $x \in \bigcup_k \left(\bigcap_{n \geq k} A_n \right)^0$. Similarly if $h_0(x) = 0$ then $x \in E^c$ if and only if $x \in \bigcup_k \left(\bigcap_{n \geq k} A_n^c \right)^0$.

From this we get

$$E = \left[\bigcup_k \left(\bigcap_{n \geq k} A_n^c \right)^0 \cap A_0^c \right]^c - \left[\bigcup_k \left(\bigcap_{n \geq k} A_n \right)^0 \cap A_0 \right]$$

Using the identity $\left[\bigcup_k \left(\bigcap_{n \geq k} A_n^c \right)^0 \right]^c = \bigcap_k \left(\bigcup_{n \geq k} A_n \right)^-$ and a little set algebra converts the above to

$$E = \left[A_0 - \bigcup_k \left(\bigcap_{n \geq k} A_n \right)^0 \right] \cup \left[\bigcap_k \left(\bigcup_{n \geq k} A_n \right)^- - A_0 \right]$$

$$\cup \left[\bigcap_k \left(\bigcup_{n \geq k} A_n \right)^- - \bigcup_k \left(\bigcap_{n \geq k} A_n \right)^0 \right]$$

Because the two unsightly terms in the above expression are similar to the ordinary limsup and liminf for sets we will introduce the following notation:

$$\text{LIMSP } A_n = \bigcap_k \left(\bigcup_{n \geq k} A_n \right)^- \qquad \text{LIMNF } A_n = \bigcup_k \left(\bigcap_{n \geq k} A_n \right)^0$$

In this notation the conditions to be satisfied for $P_0(E) = 0$ are (a) $P_0(A_0 \triangle \text{LIMNF } A_n) = 0$ and (b) $P_0(\text{LIMSP } A_n - \text{LIMNF } A_n) = 0$. From From Theorem 1 we have that if (a) and (b) hold then $P_n(A_n) \to P(\text{LIMNF } A_n) = P(\text{LIMSP } A_n)$ so we have proved the following result.

Theorem 2. If $P(\text{LIMSP } A_n - \text{LIMNF } A_n) = 0$, $P(\text{LIMNF } A_n) > 0$ and $P(A \triangle \text{LIMNF } A_n) = 0$ then $P_n(A_n) \to P(A)$ and $P_n(\cdot | A_n) \Rightarrow P(\cdot | A)$.

A special case of Theorem 2 which we will need in Sections 3 and 4 is the following:

Example: Let $S = D$ and $A_n = \{f: \inf_{s \le t_n} f(s) > 0\}$ with $t_n \to t > 0$. If $q_n = \sup_{m \ge n} t_m$ and $r_n = \inf_{m \ge n} t_m$ then

$$\begin{aligned}
\text{LIMSP } A_n &= \bigcap_{n=1}^{\infty} (\{f: \inf_{s \le r_n} f(s) > 0\})^- \\
&= \bigcap_{n=1}^{\infty} \{f: \inf_{s < r_n} f(s) \ge 0\} \\
&= \{f: \inf_{s < t} f(s) \ge 0\}
\end{aligned}$$

To compute $\text{LIMNF } A_n$ we observe

$$\bigcap_{n=m}^{\infty} A_n = \begin{cases} \{f: \inf_{s \le t} f(s) > 0\} & \text{if } t_n \ge t \text{ for some } n \ge m \\ \bigcap_{\varepsilon > 0} \{f: \inf_{s < t-\varepsilon} f(s) > 0\} & \text{if } t_n < t \text{ for all } n \ge m \end{cases}$$

Since the interior of the second set is the first, we have

$$\text{LIMNF } A_n = \bigcup_{n=1}^{\infty} \{f: \inf_{s \le q_n} f(s) > 0\} = \{f: \inf_{s \le t} f(s) > 0\}$$

and

$$\text{LIMSP } A_n - \text{LIMNF } A_n = \{f: \inf_{s \le t} f(s) = 0\} \cup \{T_0 = t\} .$$

Using Theorem 2.2 now gives that we have convergence whenever $P\{T_0 > t\} > 0$ and the two sets in the last equality above have probability zero.

This result is sufficient for most, but not all, of our desired applications. If $P\{f:f \geq 0\} = 1$ then $P\{f:\inf_{s\leq t} f(s) = 0\} = P\{T_0 \leq t\}$ and from the computations above we see that Theorem 2 can only be applied in the trivial case $P\{T_0 > t\} = 1$. To obtain our results $P\{f:f \geq 0\} = 1$ and $P\{T_0 > t\} \in (0,1)$ we will use the following.

<u>Theorem 3</u>. Let P be a probability measure and A_n be a sequence of events. If (i) there exist $G_m \uparrow A$ such that for each m $P(\partial G_m) = 0$ *and there is a* k *(depending upon* m*) so that* $A_n \supset G_m$ for all $n \geq k$, (ii) $P_n \Rightarrow P$, and (iii) $\overline{\lim_n}\, P_n(A_n) \leq P(A)$ then $P_n(A_n) \to P(A)$ and $P_n(\cdot|A_n) \Rightarrow P(\cdot|A)$.

<u>Proof</u>.

By Lemma 2 it suffices to check that $P_n(B \cap A_n) \to P(B \cap A)$ for all B with $P(\partial B) = 0$. From (i)

$$\underline{\lim}_{n\to\infty} P_n(B \cap A_n) \geq \underline{\lim}_{n\to\infty} P_n(B \cap G_m)$$

Since $P(\partial(B \cap G_m)) \leq P(\partial B) + P(\partial G_m) = 0$

$$\lim_{n\to\infty} P_n(B \cap G_m) = P(B \cap G_m)$$

Letting $m \to \infty$ now gives $\underline{\lim}_{n\to\infty} P_n(B \cap A_n) \geq P(B \cap A)$. Since $\partial(B^c) = \partial B$, $P(\partial(B^c)) = 0$ and we have

$$\underline{\lim_{n\to\infty}} P_n(B^c \cap A_n) \geq P(B^c \cap A)$$

Using (iii) now gives

$$\overline{\lim_{n\to\infty}} P_n(B \cap A_n) \leq \overline{\lim_{n\to\infty}} P_n(A_n) - \underline{\lim_{n\to\infty}} P_n(B^c \cap A_n) \leq P(B \cap A)$$

which completes the proof.

Condition (iii) suggests that to apply this theorem to examples we would like to construct the largest A for which there is a sequence $G_m \uparrow A$ which satisfies (i). To do this we observe that if G_m satisfies (i) then $G_m \subset \bigcap_{n \geq k(m)} A_n$ and $P(\partial G_m) = 0$ so there are $\varepsilon_m \downarrow 0$ so that

$$G_m^* = \{y: \{x: \rho(x,y) < \varepsilon_m\} \subset \bigcap_{n\geq k(m)} A_n\}$$

has $P(\bigcup_m G_m - \bigcup_m G_m^*) = 0$.

The sets G_m^* may have $P(\partial G_m^*) > 0$ but this is no problem. If H is any subset, $H^{\varepsilon} = \{y: \{x: \rho(x,y) < \varepsilon\} \subset k\}$ and $\varepsilon_1 < \varepsilon_2$ then $\partial(H^{\varepsilon_1}) \subseteq (H^{\varepsilon_2})^{\circ}$ so $\partial(H^{\varepsilon_1}) \cap \partial(H^{\varepsilon_2}) = \emptyset$. From this it follows that $P(\partial H^{\varepsilon}) > 0$ for only a countable number of ε, so we can pick another sequence $\varepsilon_m' \leq \varepsilon_m$ for which the associ ed G_m^* have $P(\partial G_m^*) = 0$.

From this construction we see

$$A = \sup_{\varepsilon_m \downarrow 0} \bigcup_m \left(\bigcap_{n\geq m} A_n\right)^{\varepsilon_m} = \bigcup_m \left(\bigcap_{n\geq m} A_n\right)^0 = \text{LIMNF } A_n$$

is the largest set which can occur in (i). Using this observation we can write the result of Theorem 3 in a simpler form.

<u>Theorem 4</u>. If $P_n \Rightarrow P$ and $\overline{\lim_n}\, P_n(A_n) \le P(\text{LIMNF } A_n)$ then $P_n(A_n) \to P(A)$ and $P_n(\cdot|A_n) \Rightarrow P(\cdot|A)$.

If $A_n = \{f : \inf_{s \le t_n} f(s) > 0\}$, then $\text{LIMNF } A_n = \{f : \inf_{s\le t} f(s) > 0\}$ so the condition above is $\overline{\lim_n}\, P_n\{f : \inf_{s\le t_n} f(s) > 0\} \le P\{f : \inf_{s \le t} f(s) > 0\}$. The reader should note that if $P\{f : \inf_{s\le t} f(s) > 0\} = 1$ (or $P(\text{LIMNF } A_n) = 1$ in Th. 4) then the conditional measures always converge.

Chapter 3

CONDITIONING ON $T_{(-\infty,0]} > n$

3.1 Preliminary Results

In this section we will investigate consequences of assumptions (i) and (ii). Our first result follows immediately from the uniform convergence assumed in (ii),

Theorem 1. If there is a Markov chain v_n so that $v_{[n\cdot]}/c_n$ converges to V (in the sense specified in (ii)) then V has the following weak continuity property:

$$\text{if} \quad x_n \to x, \quad \text{then} \quad V^{x_m} \Rightarrow V^x \ . \tag{1}$$

This implies, in particular, that V is a strong Markov process.

Proof.

The second fact is a well-known consequence of the first (see [21], Theorem 16.21). To prove (1) we observe that if $x_n \to x$ there is a sequence n_k increasing to ∞ so that if $y_n = x_k$ when $n_k \le n < n_{k+1}$ then $\lim_{k\to\infty} V^{x_k} = \lim_{n\to\infty} V_n^{y_n} = V^x$.

The process which can arise as limits in (ii) also have special properties because they result from scaling and contracting time in a single Markov process. The most basic of these is the scaling relationship given in the following theorem.

Theorem 2. If assumptions (i) and (ii) hold, there is a $\delta \ge 0$ so that

$$\text{for all} \quad c > 0 \quad V^{cx} \stackrel{d}{=} cV^x(\cdot c^\delta) \quad , \tag{2}$$

$$\text{for all} \quad t > 0 \quad \lim_{n\to\infty} c_{nt}/c_n \ = \ t^{1/\delta} \quad (\text{here, } t^\infty = \lim_{m\to\infty} t^m). \tag{3}$$

Proof.

Let $\lambda \in (0,1]$. Let $m_n = m_n(\lambda) = \sup\{m \le n : c_m/c_n < \lambda\}$. Since $c_{n-1}/c_n \to 1$ and $c_n \to \infty$, $c_{m_n}/c_n \to \lambda$. If $x_n \to x$ and a subsequence of m_n/n converges to $\rho \in [0,1]$, it follows from (ii) that

$$(v_{[m_n \cdot]}/c_{m_n} \mid v_0 = x_n c_{m_n}) \Rightarrow V^x$$

and a subsequence of

$$\frac{c_n}{c_{m_n}}\left(v_{\left[n\left(\frac{m_n}{n}\cdot\right)\right]}\Big/ c_n \Big| v_0 = \left(x_n \frac{c_{m_n}}{c_n}\right)c_n\right)$$

converges to $\lambda^{-1}V^{x\lambda}(\rho\cdot)$ so $V^x \overset{d}{=} \lambda^{-1}V^{x\lambda}(\rho\cdot)$.

Let x_0 be a state with $P\{V^{x_0} \equiv x_0\} < 1$. If m_n/n has two subsequential limits $\rho_1, \rho_2 \in [0,1]$ with $\rho_1 < \rho_2$ then

$$\lambda^{-1}V^{x_0}(\rho_1\cdot) \overset{d}{=} V^{x_0/\lambda} \overset{d}{=} \lambda^{-1}V^{x_0}(\rho_2\cdot)$$

so if $t > 0$ and n is a positive integer $V^{x_0}(t) \overset{d}{=} V^{x_0}\left(t\left(\frac{\rho_1}{\rho_2}\right)^n\right)$.

Letting $n \to \infty$ and using the right continuity of V^{x_0} at 0 gives $P\{V^{x_0}(t) = x_0\} = 1$ for each t, a contradiction, so $\lim_{n\to\infty} m_n(\lambda)/n$ exists and is positive.

If we let $\rho(\lambda) = \lim_{n\to\infty} m_n(\lambda)/n$ then ρ is a positive nondecreasing function which satisfies $\rho(s)\rho(t) = \rho(st)$. From this it is immediate that $\rho(s) = s^d$ for some $\delta \ge 0$ and (2) holds.

To prove (3) we will consider two cases. First, let $\delta > 0$. If $\lambda_1^\delta < t < \lambda_2^\delta$ then for n sufficiently large $m_n(\lambda_1) < nt < m_n(\lambda_2)$

so $\lambda_1 \le \underline{\lim}_{m}\, c_{nt}/c_n \le \overline{\lim}_{n}\, c_{nt}/c_n \le \lambda_2$. Since this holds for all λ_1 and λ_2 with $\lambda_1^{\delta} < t < \lambda_2^{\delta}$ this means $\lim_{n\to\infty} c_{nt}/c_n = t^{1/\delta}$. If $\delta = 0$ a similar argument shows $\overline{\lim}_{n\to\infty}\, c_{nt}/c_n < \varepsilon$ for all $\varepsilon > 0$ and this completes the proof.

<u>Remark</u>. A function L is slowly varying if $\lim_{t\to\infty} L(xt)/L(t) = 1$ for all $x > 0$. Using this notation conclusion (3) can be written as $c_n = n^{1/\delta}L(n)$. Since we will write many statements like this in what follows we will use the letter L to denote slowly varying functions. The value of $L(n)$ is rarely important for our arguments and in general will change from line to line. Subscripts and other ornaments will be attached when we want to emphasize that the slowly varying function depends upon the indicated parameters.

If $\delta > 0$ we can rewrite (2) as

$$V^{x} \stackrel{d}{=} n^{-1/\delta} V^{xn^{1/\delta}}(n\cdot) \tag{4}$$

so (1) and (2) characterize the processes which can occur as limits in (ii). If $\delta = 0$, however, (2) becomes $V^{cx} \stackrel{d}{=} cV^{x}$ and we can no longer guarantee that there are $c_n \to \infty$ so that $c_n^{-1} V^{c_n x_n}(n\cdot)$ converges. We have not been able to characterize the limits which can occur when $\delta = 0$. The next few results show that these processes have some strange properties.

An immediate consequence of Theorem 1 is the fact that for all $c > 0$

$$P^{cx}\{T_0 > t\} = P^{x}\{T_0 > tc^{\delta}\} \tag{5}$$

If $\delta = 0$ this means that $P^y\{T_0 > t\}$ has the same value for all $y > 0$ so using the strong Markov property

$$P^y\{T_0 > s+t\} = E^y[T_0 > s;\ P^{X(s)}\{T_0 > t\}]$$
$$= P^y\{T_0 > s\}P^y\{T_0 > t\} \quad .$$

Since $\phi(t) = P^y\{T_0 > t\}$ is nonincreasing, nonnegative, and satisfies $\phi(t+s) = \phi(s)\phi(t)$ this means $P^y\{T_0 > t\} = e^{-\lambda t}$ for some $\lambda \geq 0$ (which is independent of y).

This shows that (iii) is always satisfied if $\delta = 0$. If $\delta > 0$, however, we are not so lucky. In this case taking $c > 1$ in (5) gives only an inequality:

$$P^x\{T_0 > t\} \geq P^y\{T_0 > t\} \quad \text{when} \quad x \geq y > 0 \qquad (6)$$

so we are forced to take a new approach.

Let $S_x = \inf\{t: P^x\{T_0 > t\} = 0\}$. What we would like to show is: $S_x = \infty$ for each $x > 0$. From (2), we have:

$$\text{if} \quad c > 0 \qquad S_{cy} = c^{\delta} S_y \qquad (7)$$

so either all the S_x are infinite or none is.

Suppose $S_y < \infty$. Using the strong Markov property

$$0 = P^y\{T_0 > S_y\} = E^y[T_{y+\varepsilon} < T_0; P^{V(T_{y+\varepsilon})}\{T_0 > S_y - T_{y+\varepsilon}\}]$$

Since $V^y(T_{y+\varepsilon}) \geq y + \varepsilon$ and $S_y - T_{y+\varepsilon} < S_y$ it follows from (7) that the integrand is positive so $P^y\{T_{y+\varepsilon} < T_0\} = 0$ for each $\varepsilon > 0$.

Since V is a strong Markov process this implies $V^y(t \wedge T_0)$ is

nonincreasing. When we note that for each $t > 0$,

$0 = P^y\{T_0 > S_y\} \geq P\{V(t) = y, T_0 > t\}P^y\{T_0 > S_y - t\}$ we have shown:

$$\text{if } S_y < \infty, \quad V^y(t) \text{ is strictly decreasing for } t < T_0 \tag{8}$$

Having arrived at a strange conclusion under the assumption $S_y < \infty$ we might hope to continue and derive a contradiction. The next example shows that this is not possible.

Example. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with mean $\mu < 0$. If $S_n = S_{n-1} + X_n$ for $n \geq 1$ then $S_{[n\cdot]}/n$ converges in the Markov sense to "uniform motion to the left at rate $-\mu$" (see [22], Exercise 3.7 if you need a more precise description). For this limit $P^y\{T_0 > t\} = 0$ if $y + \mu t \leq 0$ so $S_y = -y/\mu$.

In this example the limit is degenerate so we wonder: Are there nontrivial limits with $S_y < \infty$?

We will show in Section 4 that no process with this property occurs as a limit for any of the examples we consider, but the question of whether (i) and (ii) are sufficient to guarantee (iii) has not been resolved. The solution of this problem is really of minor importance for the applications; it is usually very easy to use (8) to verify (iii).

Up to this point we have only used the scaling relationship for $x > 0$. If we let $x = 0$ in (2) and (5) then we get two more formulas to help us analyze the limit process.

$$V^0 \overset{d}{=} cV^0(\cdot c^{\delta}) \tag{9}$$

$$P^0\{T_0 > t\} = P^0\{T_0 > tc^{\delta}\} \tag{10}$$

If $\delta = 0$, (9) says $V^0 \overset{d}{=} cV^0$ for all $c > 0$ so $V^0 \equiv 0$. Combining this result with the fact that $P^x\{T_0 > t\} = e^{-\lambda t}$ for $x > 0$ gives

$$\lim_{\varepsilon \downarrow 0} P\left(\sup_{0 \le t \le 1} V^{\varepsilon}(t) > \delta \mid T_0 > 1\right)$$

$$\le e^{\lambda} \lim_{\varepsilon \downarrow 0} P\left\{\sup_{0 \le t \le 1} V^{\varepsilon}(t) > \delta\right\} = 0$$

so $(V^{\varepsilon} \mid T_0 > 1) \Rightarrow 0$ as $\varepsilon \downarrow 0$. Taking a peak ahead into Section 3.3 we see that this means the only possible limit of V_n^+ is 0 so we will abandon this case and label it trivial.

If $\delta > 0$, (10) shows that $P^0\{T_0 > t\}$ has the same value for all $t > 0$. Since $P^0\{T_0 > 0\} = \lim_{u \downarrow 0} P^0\{T_0 > u\}$ it follows from the Blumenthal 0-1 law ([22], Theorem 5.17) that

$$P^0\{T_0 > t\} \text{ is either } \equiv 0 \text{ or } \equiv 1. \tag{11}$$

Since $\{T_0 > t\}$ is open, using (5), (1) and Theorem 1.1 gives

$$P^x\{T_0 > t\} \ge \lim_{y \downarrow 0} P^y\{T_0 > t\} \ge P^0\{T_0 > t\} \tag{12}$$

for all $t, x > 0$.

From (12) we see that if $P^0\{T_0 > t\} \equiv 1$ then $P^x\{T_0 > t\} = 1$ for all $t, x > 0$ and so we expect that the conditioning to stay positive will have no effect. For positive levels this is a consequence of the

results of Chapter 2: if $x_n \to x > 0$ using Theorem 2.4 gives $(V^{x_n} | N > n) \Rightarrow (V^x | T_0 > 1) \Rightarrow V^x$.

If $x_n \to 0$ the situation becomes more complicated. If $\underline{\lim}_n P_n^{x_n}\{N > n\} < 1$ then we cannot apply the results of Chapter 2 (each theorem has $P_n(A_n) \to P(A)$ as a conclusion) and if $\underline{\lim}_n P_n^{x_n}\{N > n\} = 0$, V_n^+ may fail to be tight. Conditions for convergence in this case will be given in Section 3.3. The results given there will show that if the limit exists in the sense of (a) then $V_n^+ \Rightarrow V^0$, i.e., the conditioning has no effect.

For the rest of the paper we will be mainly concerned with what happens when $P^x\{T_0 > t\} \not\equiv 1$ for some (and hence all) $x > 0$. Since $P^x\{T_0 > t\}$ is decreasing $\lim_{t\to\infty} P^x\{T_0 > t\}$ exists for each $x > 0$. Using the scaling relationship gives that this limit is independent of x. Call it λ. From the Markov property

$$P^x\{T_0 > t+s\} = E^x\{T_0 > t;\ P^{V(t)}\{T_0 > s\}\}$$

Letting $s \to \infty$ gives $\lambda = \lambda P^x\{T_0 > t\}$ so $\lambda = 0$.

If $\delta = 0$, this agrees with our previous calculation. If $\delta > 0$, we can use (4) to conclude

$$\lim_{x\downarrow 0} P^x\{T_0 > t\} = \lim_{u\uparrow\infty} P^1\{T_0 > u\} = 0 \quad \text{iff} \quad P^x\{T_0 > t\} \not\equiv 1 . \qquad (13)$$

The reason for interest in this conclusion is the following:

$$\left.\begin{array}{l} \text{Suppose } \lim_{x\downarrow 0} P^x\{T_0 > t\} = 0 \text{ for all } t > 0 \text{ and (iv) holds.} \\ \text{If for each } m \;\; P\{N > m \mid v_0 = x\} \text{ is an increasing function of} \\ x \text{ then} \\ \qquad \text{(v)} \;\; P^{x_n}\{N > nt_n\} \to 0 \text{ whenever } x_n \to 0 \text{ and } t_n \to t > 0. \end{array}\right\} \quad (14)$$

There is a converse to this proved in [41]:

$$\text{if (v) holds then so does (iv)}\,. \tag{15}$$

Since it is usually more difficult to verify (v) than (iv), (15) is not a useful result for checking that (iv) holds. To obtain the results we will use to check (iv) in Chapter 4 we will use the results of Chapter 2.

If $P^x\{T_0 = t\} = 0$ and $P^0\{T_0^- = 0\} = 1$ then from the strong Markov property $P^x\{f: \inf_{0\le s\le t} f(s) = 0\} = 0$ so using Theorem 2.2 gives $(V_n^{x_n} | N > nt_n) \Rightarrow (V^x | T_0 > t)$ whenever $x_n \to x > 0$ and $t_n \to t > 0$. From (9)

$$P^0\{T_0^- = 0\} \ge \lim_{t\to 0} P\{V^0(t) < 0\} = P\{V^0(1) < 0\} \tag{16}$$

so if $P^0\{V^0(1) < 0\} > 0$ using the Blumenthal 0-1 law gives $P^0\{T_0^- = 0\} = 1$ and the result above can be applied to conclude:

$$\text{if } P^x\{T_0 = t\} = 0 \text{ and } P\{V^0(1) < 0\} > 0, \text{ (iv) holds.} \tag{17}$$

On the other hand, if $P\{V^0(1) < 0\} = 0$

$$P\{\inf_{0\le s\le t} V^0(s) \ge 0\} \ge 1 - \sum_{q,\text{rational}} P\{V^0(q) < 0\} = 1 \tag{18}$$

so $V^0 \ge 0$ and Theorem 2.2 cannot be applied. In this case we will use Theorem 2.4 or another trick (see Section 4.4).

3.2 Conditions for Tightness*

According to Theorem 15.2 in [20], a sequence of probability measures on D is tight if and only if the following two conditions hold:

(a) $\lim\limits_{M\to\infty} \overline{\lim\limits_{n\to\infty}}\, P_n\{f: \sup\limits_t |f(t)| > M\} = 0$

(b) if $\omega_f'(\delta)$ is the quantity defined by (5) of Section 1.2 then for each $\varepsilon > 0$

$$\lim_{\delta\to\infty} \overline{\lim_{n\to\infty}}\, P_n\{f: \omega_f'(\delta) < \varepsilon\} = 0$$

Because of the complexity of the definition of ω_f' the second condition is usually difficult to verify. In this section we will assume (i)-(iv) hold and develop equivalent conditions, which are easier to check in our special case, by examining the behavior of the path before and after hitting $[\varepsilon,\infty)$.

If $T_\varepsilon(f) > \delta$ we can let $t_1 = T_\varepsilon(f)$ in the definition of ω_f' and obtain

$$\omega_f'(\delta) \le \varepsilon \vee \omega_f'(\delta; T_\varepsilon, 1) \tag{1}$$

When $f = V_n^+$ the last expression is the "D modulus of continuity" of a process which starts from a height $V_n^+(T_\varepsilon \wedge 1)$ and is conditioned to stay positive for $(1-T_\varepsilon)^+$ time units. Since we have assumed (iv), the results of Section 2 show that $(V_n^{x_n} | N > n) \Rightarrow (V^x | T_0 > 1)$ when $x_n \to x > 0$ and using the inequality above we can prove the following.

---

*Note: Throughout this section we will assume that δ, the exponent in (2) of section 3.1, is positive.

Theorem 3. V_n^+ is tight if and only if the following two conditions hold

(3a) for some $\varepsilon > 0$ $\lim_{M\to\infty} \overline{\lim_{n\to\infty}} P\{V_n^+(T_\varepsilon) > M\} = 0$

(3b) for all $\varepsilon > 0$ $\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} P\{T_\varepsilon(V_n^+) < \delta\} = 0$

That is, we have tightness if the conditioning does not make the process jump too high or leave zero too fast.

Proof.

The conditions are necessary since they follow from (a) and (b) above. To prove sufficiency define the post-T_ε process

$$X_n^+(\cdot) = (v_{[n(T_\varepsilon+\cdot)]}/c_n \mid T_\varepsilon \leq 1,\ N > n)$$

Since v_n is a Markov chain

$$X_n^+(\cdot) \overset{d}{=} (v_{[n\cdot]}/c_n \mid v_0 = Y_n, T_0 > L_n)$$

where

$$Y_n = (v_{nT_\varepsilon}/c_n \mid T_\varepsilon \leq 1, N > n)$$

and

$$L_n = (1-T_\varepsilon \mid T_\varepsilon \leq 1, N > n) \quad .$$

From Prohorov's theorem ([20] Theorems 6.1 and 6.2) a sequence of probability measures on D is tight if and only if every subsequence has a further subsequence which converges weakly, so it is enough to show that for any subsequence (a) and (b) hold for some further subsequence.

Let $\varepsilon > 0$. If $P^+_{n_k}\{T_\varepsilon \le 1\} \to 0$ as $k \to \infty$ then (a) and (b) hold so it suffices to consider subsequences for which $\underline{\lim}_{k\to\infty} P^+_{n_k}\{T_\varepsilon \le 1\} > 0$. In this case the tightness of Y_{n_k} follows from (3a). Since $0 \le L_n \le 1$, (Y_{n_k}, L_{n_k}) is tight and so there is a sequence of integers $m_j = n_{k_j} \uparrow \infty$ so that $(Y_{m_j}, L_{m_j}) \Rightarrow (Y,L)$.

Let h be a bounded continuous function from D to R. If $g_n(x,t) = E(h(V^x_n) | T_0 > t)$ then $E(h(X^+_n)) = E(g_n(Y_n, L_n))$. Using (iv) and the results of Section 2 we have that as $x_n \to x > 0$ and $t_n \to t \ge 0$

$$g_n(x_n, t_n) \to g(x,t) = E(h(V) | V(0) = x, T_0 > t)$$

so from Theorem 2.1 $Eh(X^+_{m_k}) \to Eg(Y,L)$. From this we can conclude $X^+_{m_k} \Rightarrow (V | V(0) = Y, T_0 > L)$, a process we will denote by V^*.

Since $X^+_{m_k} \Rightarrow V^*$ using Theorem 1.1 gives that $\overline{\lim}_k Eh(X^+_{m_k}) \le Eh(V^*)$ when h is bounded and upper semicontinuous. Applying this result with $h(f) = 1 \wedge (\sup_t f(t) - (M-1))^+$ (see (3) of Section 1.2) and $h(f) = \omega'_f(\delta) \wedge 1$ (see Theorem 1.2) and using the obvious inequalities

$$\sup_t f(t) \le \varepsilon \vee \sup_{t \ge T_\varepsilon} f(t)$$

$$P^+_n\{\omega'_f(\delta) > \varepsilon\} \le P^+_n\{T_\varepsilon < \delta\} + P^+_n\{\omega'_f(\delta; T_\varepsilon, 1) > \varepsilon | T_\varepsilon \le 1\}$$

completes the proof.

Condition (3a) may be difficult to check directly because it involves estimating the value of V_n^+ at a random time. Using the scaling relationship and the Markov property we have for $t < 1$ that

$$P\{V(1) > K \mid V(t) = x\} = P\{V^x(1-t) > K\}$$

$$= P\{xV^1((1-t)x^{-\delta}) > K\}$$

If $\delta > 0$ then from the right continuity of V^1 as $x \to \infty$ the above converges to 1 uniformly for $t \in [0,1]$ so

$$\lim_{M\to\infty} P\{V(1) > K \mid V(T_\varepsilon) > M\} = 1$$

From scaling and the right continuity of V^1

$$\lim_{x\uparrow\infty} P^x\{T_0 > 1\} = \lim_{t\downarrow 0} P^1\{T_0 > t\} = 1$$

so the same statement holds for the process V^+. This suggests:

<u>Theorem 4</u>. A sufficient condition for (3a) is

$$\lim_{K\to\infty} \overline{\lim_{n\to\infty}} P\{V_n^+(1) > K\} = 0$$

<u>Remark</u>. From (a) it is clear that this is necessary for tightness. An argument similar to that given in the proof below will show that this is necessary for (3a).

<u>Proof</u>.

Using the Markov property, if $\varepsilon < K$

$$P\{V_n^+(1) > K\} = E[T_\varepsilon \le 1; q_K^n(V_n^+(T_\varepsilon), 1-T_\varepsilon)]$$

where $q^n_K(x,t) = P(V_n(1) > K \mid V_n(1-t) = x, T_0 > 1)$. From (iv) and Theorem 1.1, if $x_n \to x > 0$ and $t_n \to t \geq 0$

$$\overline{\lim_{n\to\infty}}\ q^n_K(x_n,t_n) \geq q(x,t)$$

where $q_K(x,t) = P(V(1) > K \mid V(1-t) = x, T_0 > 1)$ so for $u \leq 1$

$$\overline{\lim_{n\to\infty}}\ P\{V^+_n(1) > K\} \geq \overline{\lim_{n\to\infty}}\ E[V^+_n(T_\varepsilon) > 2Ku^{-1/\delta}; q^n_K(V^+_n(T_\varepsilon), 1-T_\varepsilon)]$$

$$\geq [\inf\{q_K(x,s) : x \geq 2Ku^{-1/\delta}, 0 \leq s \leq 1\}\ \overline{\lim_{n\to\infty}}\ P\{V^+_n(T_\varepsilon) > 2Ku^{-1/\delta}\}$$

From scaling $q_K(x,t) = q_{Kc}(xc, tc^\delta)$ so if $2K/x \leq 1$, $q_K(x,t) \geq q_K(2K, t(2K/x)^\delta)$ and from above

$$\overline{\lim_{n\to\infty}}\ P\{V^+_n(1) > K\} \geq \left[\inf_{0\leq s\leq u} q_K(2K,s)\right]\ \overline{\lim_{n\to\infty}}\ P\{V^+_n(T_\varepsilon) > 2Ku^{-1/\delta}\}$$

Now

$$1 \geq q_K(2K,s) \geq \frac{P(V_n(s) > K \mid V_n(0) = 2K) - P(T_0 \leq s \mid V_n(0) = 2K)}{P(T_0 > s \mid V_n(0) = 2K)}$$

Letting $u \to 0$ gives

$$\overline{\lim_{n\to\infty}}\ P\{V^+_n(1) \geq K\} \geq \lim_{M\to\infty}\ \overline{\lim_{n\to\infty}}\ P\{V^+_n(T_\varepsilon) > M\}$$

and letting $K \to \infty$ gives the desired result.

From Theorem 5 if we know that $V^+_n(1)$ converges then (3a) is satisfied. The next theorem gives a sufficient condition for (3b).

Theorem 5. Let P_n^* be the probability measures induced on $D[-1,1]$ by $V_n^+(t \vee 0)$. If (3a) holds $\{P_n^*, n \geq 1\}$ is tight. If, in addition, for every P^* which is the limit of a subsequence $P_{n_k}^*$ we have $P^*\{f: f(0) \neq f(0-)\} = 0$ then $\{P_n^+, n \geq 1\}$ is tight.

Proof.

For all $f \in D[-1,1]$ which are constant on $[-1,0)$ if $\delta < 1$ we have

$$\omega_f'(\delta;-1,1) \leq \varepsilon \vee \omega_f'(\delta;T_\varepsilon,1)$$

From this

$$P_n^*\{\omega_f'(\delta;-1,1) > \varepsilon\} \leq P_n^*\{\omega_f'(\delta;T_\varepsilon,1) > \varepsilon\}$$

so using the proof of Theorem 3 we see that (3a) is sufficient for tightness in $D[-1,1]$.

To prove the other result we note that by Prohorov's theorem it is sufficient to show that if $P_{n_k}^* \Rightarrow P^*$ then $P_{n_k}^+ \Rightarrow P^+ = P^*\pi^{-1}$ where π is the natural projection from $D[-1,1]$ to $D[0,1]$. If $h: D[0,1] \to R$ has $P^+(\Delta_h) = 0$ where Δ_h is the set of discontinuities of h then $P^*\{f: f(0) \neq f(0-) = 0\}$ implies that $P^*(\Delta_{h\circ\pi}) = 0$. The desired result now follows from Billingsley's form of the continuous mapping theorem ([20], Theorem 5.2): $P_n \Rightarrow P$ if and only if $P_n h^{-1} \Rightarrow Ph^{-1}$ for each measureable real valued function h with $P(\Delta_h) = 0$.

Combining the conclusions of Theorems 3,4, and 5 gives the following result.

Theorem 6. V_n^+ is tight if and only if

(6a) $\lim_{K\to\infty} \overline{\lim}_{n\to\infty} P\{V_n^+(1) > K\} = 0$

(6b) $\lim_{t\to 0} \overline{\lim}_{n\to\infty} P\{V_n^+(t) > \delta\} = 0$ for each $\delta > 0$.

From Theorem 6 if we know that the finite dimensional distributions of V_n^+ converge to those of a process V^* with $P\{V^*(0) = 0\} = 1$, then the sequence is tight.

In Theorem 10 below we will give conditions which imply that if V_n^+ is tight then the limit is $\lim_{x\downarrow 0} (V^x | T_0 > 1)$ (assuming this exists) so in cases when the convergence of finite dimensional distributions is not known we would like to check that the sequence is tight without computing the limit of the distributions.

One way of doing this (which we will use in Section 4.3) is to observe that if $V_{n_k}^+ (t \vee 0)$ converges almost surely (as a sequence of random elements of $D[-1,1]$) to a process V^* with $P\{V^*(0) > 2\delta\} = \rho > 0$ for some $\delta > 0$, then from the definition of the metric for $D[-1,1]$, $\underline{\lim}_{k} P\{V_{n_k}^+(T_\delta) - \delta > \delta\} \geq \rho$. Using Theorem 5 and Lemma 2.1 now gives:

Theorem 7. If for each $\varepsilon > 0$ $(V_n^+(T_\varepsilon) - \varepsilon)^+ \Rightarrow 0$ then V_n^+ is tight.

3.3 Convergence of Finite Dimensional Distributions

In this section we will assume V_n^+ is tight and derive conditions for V_n^+ to converge. Our method of proof is not the usual one suggested by the title of this section, however. We will prove convergence by showing that all convergent subsequences have the same limit.

The first step is to consider what processes can occur as limits of the V_n^+. From (i)-(iv) and the results of Section 2, if $x_n \to x > 0$ $(V_n^{x_n} | N > n) \Rightarrow (V^x | T_0 > 1)$. Letting x_n go to zero very slowly we see that if V_n^+ converges for all $x_n \to 0$ then $\lim_{x \downarrow 0} (V^x | T_0 > 1)$ exists and is the limit process for any $x_n \to 0$.

Assuming $\lim_{x \downarrow 0} (V^x | T_0 > 1)$ exists and writing $(V^0 | T_0 > t)$ for $\lim_{x \downarrow 0} (V^x | T_0 > t)$ we can give a simple formula for the processes which can occur as limits of subsequences of V_n^+.

<u>Theorem 8</u>. If $V_{n_k}^+ \Rightarrow V^*$ then there are random variables $t^* \in [0,1]$ and $x^* \geq 0$ with $P\{t^* = 0, x^* > 0\} = 0$ so that

$$V^*(\cdot) \overset{d}{=} 1_{\{t^* \leq \cdot\}} (V^{x^*}(\cdot - t^*) | T_0 > 1 - t^*) \tag{1}$$

<u>Remark</u>. This characterization shows that if $(V^\varepsilon | T_0 > 1) \Rightarrow 0$ as $\varepsilon \to 0$ then 0 is the only possible limit.

<u>Proof</u>.

From the proof of Theorem 3.3 $V^*(T_\varepsilon(V^*)+t)$ behaves like V starting from $V^*(T_\varepsilon)$ and conditioned to stay positive for $1-T_\varepsilon(V^*)$

units of time. As ε decreases, $T_\varepsilon(V^*)$ does not increase so as $\varepsilon \downarrow 0$, $T_\varepsilon(V^*)$ converges to a limit t^*. Since V^* is right continuous this means $V^*(T_\varepsilon)$ converges to a limit x^*.

Under the hypothesis of Theorem 8, $(x,t) \to (V^x | T_0 > t)$ is a continuous function from $[0,\infty)x(0,\infty)$ to $D[0,1]$ so using the continuous mapping theorem we see that $V^*(T_\varepsilon(V^*)+t) \Rightarrow (V^{x^*} | T_0 > 1-t^*)$. Since $0 \le V^* < \varepsilon$ on $[0,T_\varepsilon(V^*))$ this shows V^* has the representation given by (1).

To see that $P\{t^* = 0, x^* > 0\} = 0$ observe that since $V^+_{n_k} \Rightarrow V^*$ in D, $x_{n_k} = V^+_{n_k}(0) \Rightarrow V^*(0)$ so $V^*(0) = 0$.

It is easy but tedious to show that all the processes given by formula (1) are possible limits. In the next example we show how to do this if $P\{t^* = 0\} = 1-p$ and $P\{t^* = t, x^* \le x\} = pF(x)$ where $p,t \in (0,1]$ and F is a distribution with $F(0-) = 0$. We leave the general construction to the reader's imagination.

Example. Let v_n be an integer valued Markov chain (say, a Bernoulli random walk) with satisfies (i)-(iv) and has $V^+_n \Rightarrow V^+$ for all $x_n \to 0$.

Let $\{r_{ij}, 0 \le i \le j < \infty\}$ be a collection of distinct numbers taken from $(0,1)$ and let $\lambda_j \downarrow 0$. Let v'_n be the Markov chain defined on $\mathbb{Z} \cup \{r_{ij}, 0 \le i \le j < \infty\}$ which makes the same transitions on the integers as v_n and is defined on the other states by the following rules

if $i = 0$ $\qquad P\{v_1' = 1 \quad | v_0' = r_{0,j}\} = 1-p$

$$P\{v_1' = r_{1,j} \,|\, v_0' = r_{0,j}\} = p$$

if $0 < i \neq [jt]-1$ $\qquad P\{v_1' = 0 \quad | v_0' = r_{ij}\} = 1-\lambda_j$

$$P\{v_1' = r_{i+1,j} | v_0' = r_{ij}\} = \lambda_j$$

if $i = [jt]-1$ $\qquad P\{v_1' = 0 \quad | v_0' = r_{ij}\} = 1-\lambda_j$

$$P\{v_1' = e_{h,j} \,|\, v_0' = r_{ij}\} = \lambda_j \rho_{hj}$$

where for each j, e_{hj} $h \geq 1$ is an increasing sequence of positive integers and ρ_{hj} $h \geq 1$ is a nonnegative sequence with $\sum_{h\geq 1} \rho_{h,j} = 1$ so that if

$$F_j(t) = \sum_{h, e_{hj} \leq c_n t} \rho_{hj}$$

then $F_j \Rightarrow F$ as $j \to \infty$.

Having identified the possible limits of subsequences of $(V_n^{x_n} | T_0 > 1)$ the next step in solving problem (a) is to determine for which V^* there is a Markov chain v_n so that $(V_n^{x_n} | T_0 > 1) \Rightarrow V^*$ for all $x_n \to 0$.

If $\overline{\lim_r} \, P_n^{x_n}\{N > n\} > 0$ for some $x_n \to 0$ then it is easy to show that a subsequence of $V_{n_k}^+$ converges to V^0 so in this case if the convergence takes place in the sense of (a) the conditioning will have no effect.

To characterize the limits which can occur when (v) holds we will investigate the convergence in the case $x_n c_n \equiv a$. In this instance the limit process results from conditioning and scaling a single sequence of random variables so there is a scaling relationship which allows us to compute the distribution of V^* from that of $V^*(1)$.

<u>Theorem 9</u>. Let $x_n c_n \equiv a$, $Q^a(\cdot) = P(\cdot \mid v_0 = a)$. If $V_n^+(1) \Rightarrow 0$ then V_n^+ converges to a process which is $\equiv 0$. If $V_n^+(1) \Rightarrow v^*$ with $P\{v^* = 0\} < 1$ then $Q^a\{N > n\} = n^{-\beta} L_a(n)$. In the second case if (v) holds then the finite dimensional distributions of $\{V_n^+(s),\ 0 < s \le 1\}$ converge to those of a nonhomogeneous Markov process V^+ which has

$$P(V^+(t) \in dy) = t^{-\beta} P(t^\gamma v^* \in dy) P^y\{T_0 > 1-t\} \qquad (2)^*$$

and

$$P(V^+(t) \in dy \mid V^+(s) = x) = \frac{P(V^x(t-s) \in dy, T_0 > t-s) P^y\{T_0 > 1-t\}}{P^x\{T_0 > 1-s\}}$$

for $x > 0$.(3)

If $V_n^+(t) \Rightarrow 0$ as $t \to 0$ then V_n^+ is tight and $V_n^+ \Rightarrow V^+$.

<u>Proof</u>.

The first result is obvious: observe that if V^* is given by (1) then $P(V^*(t+s) > 0 \mid V^*(t) = x) = P(V^x(s) > 0 \mid T_0 > 1-t) = 1$ so V^* does not hit zero after it hits a positive level.

To prove the second statement note that if $\lambda \to 0$

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*Note: $\gamma = 1/\delta$.

$$\frac{Q^a\{N > (1+\lambda)n\}}{Q^a\{N > n\}} = \int_{(0,\infty)} Q^a(V_n(1)\epsilon dx | N > n) P(N > \lambda n | v_0 = xc_n) \tag{4}$$

and from the hypothesis as $x_n \to x \geq 0$ $\phi_n^\lambda(x_n) = P(N > \lambda n | v_0 = x_n c_n)$ converges to $P^x\{T_0 > \lambda\} = \phi^\lambda(x)$.

$\phi^\lambda(x) > 0$ for $x > 0$ so if $V_n^+(1) \Rightarrow v^*$ with $P\{v^* = 0\} < 1$ then from Theorem 2.1 $Q^a\{N > (1+\lambda)n\}/Q^a\{N > n\}$ converges to a positive limit. If we let $\rho(1+\lambda)$ denote the value of this limit then since $\rho(st) = \rho(s)\rho(t)$, ρ is measureable, and $\rho(s) \leq 1$ for $s \geq 1$ we can conclude $\rho(s) = s^{-\beta}$ for some $\beta \geq 0$.

This shows that $Q^a\{N > n\}$ has the indicated form. To prove that the finite dimensional distributions of V_n^+ converge we will use this fact and the following formula:

If $k \geq 1$, $0 < t_1, \ldots < t_k \leq 1$ and $y_1, \ldots y_k$ are positive

$$P\{V_n^+(t_1) \leq y_1 \ \ldots \ V_n^+(t_k) \leq y_k\}$$

$$= \frac{Q^a\{N > nt_1\}}{Q^a\{N > n\}} \int_{(0,y_1]} Q^a(t_1^{-\gamma} V_{nt_1}(1)\epsilon dx | N > nt_1) \psi_n^{t_1}(x) \tag{5}$$

where

$$\psi_n^{t_1} = P(V_n(t_2) \leq y_2 \ldots V_n(t_k) \leq y_k, \inf_{t_1 \leq s \leq 1} V_n(s) > 0 | V_n(t_1) = x)$$

From (iv) and the results of Chapter 2 if $x_n \to x > 0$

$$\psi_n^{t_1}(x_n) \to \psi^{t_1}(x) = P(V(t_2) \leq y_2, \ldots V(t_k) \leq y_k, \inf_{t_1 \leq s \leq 1} V(s) > 0 | V(t_1) = x)$$

whenever the y_i are all continuity points of the distributions of the

$V(t_i)$, so if we can show $P\{v^* = 0\} = 0$ we can use Theorem 2.1 to conclude

$$P(V_n^+(t_1) \le y_1 \ldots V_n^+(t_k) \le y_k) \to t_1^{-\beta} \int_{(0,y_1]} P(t_1^\gamma v^* \in dx)\psi_{t_1}(x)$$

which shows the limit process has the indicated form.

Let $G_n(x) = P\{V_n^+(t) \le x\}$, $G(x) = P\{v^* \le x\}$. From (iv) and Theorem 2.1

$$\int_{(0,\infty)} G_{n_k}(dx)\phi_{n_k}^\lambda(x) \to \int_{[0,\infty)} G(dx)\phi^\lambda(x)$$

Since $Q^a\{N > (1+\lambda)n\}/Q^a\{N > n\} \to (1+\lambda)^{-\beta}$ using (3) gives

$$(1+\lambda)^{-\beta} = \int_{[0,\infty)} G(dx)\phi^\lambda(x)$$

Now (v) implies $\phi^\lambda(0) = 0$ and we always have $\phi^\lambda(x) \le 1$ so this means that $G(0) \le 1-(1+\lambda)^{-\beta}$ for all $\lambda > 0$ or $G(0) = 0$.

To complete the proof of Theorem 9, we observe that the last statement is an immediate consequence of Theorems 3.4 and 3.5.

Combining the results of Theorems 8 and 9 we observe that if (i)-(v) hold and V_n^+ converges in the sense specified by problem (a) then the limit is either $\equiv 0$ or > 0 at each $t > 0$ so there are only two possible limits (assuming $\lim_{x\to 0} (V^x | T_0 > 1)$ exists).

At this point we are ready to consider conditions for convergence to each of these limits but there is not really much to say. The next result, which summarizes our main conclusions is an easy consequence of Theorems 8 and 9.

Theorem 10. Let v_n be a Markov chain for which (i)-(iv) hold. Let $x_n \to 0$ and suppose V_n^+ is tight. $V_n^+ \Rightarrow 0$ if and only if

$$P\left\{\max_{0\le s\le 1} V_n^+(s) > \varepsilon\right\} \to 0 \quad \text{for all } \varepsilon > 0 \tag{6}$$

If $V^+ = \lim_{x\to 0} (V^x | T_0 > 1)$ exists and is $\neq 0$ then $V_n^+ \Rightarrow V^+$ if and only if

$$\lim_{\delta\downarrow 0} \underline{\lim}_{n\to\infty} P\{V_n^+(t) > \delta\} = 1 \quad \text{for all} \quad t > 0 \tag{7}$$

If $x_n c_n \equiv a$ and (v) holds then condition (7) is equivalent to $Q^a\{N > n\} = n^{-\beta} L_a(n)$.

Proof.

The first result is trivial. To prove the last two it is sufficient to show that the condition given in each case is equivalent to assuming that for all subsequential limits V^* $P\{V^*(t) > 0\} = 1$ for all $t > 0$. For the second result this claim is obvious. For the third it follows from the last computations in the proof of Theorem 9.

Chapter 4

EXAMPLES AND EXTENSIONS

4.1 Random Walks

If $X_1, X_2 \ldots$ is a sequence of independent and identically distributed random variables, $S_n = S_{n-1} + X_n$, $n \geq 1$ defines a random walk. Necessary and sufficient conditions for the convergence of $(S_n - b_n)/a_n$ are known (cf. [29], Chapter 7). In this section we will use some of these results to show that if S_n/a_n converges in distribution to G then (i)-(iv) hold and the results of Chapter 3 can be applied to prove the appropriate conditioned limit theorems.

Theorem 1. For the nondegenerate distribution G to be the limit of some sequence of normalized sums $(S_n - b_n)/a_n$ it is necessary and sufficient that it be stable, that is, if $X, X_1, \ldots X_k$ are independent and have distribution G then there are constants $a'_k > 0$ and b'_k such that

$$X_1 + \ldots + X_k \overset{d}{=} a'_k X + b'_k \quad .$$

Theorem 2. $\phi(\theta) = Ee^{i\theta X}$ is the characteristic function of a stable law if and only if

$$\log \phi(\theta) = i\lambda\theta - c|\theta|^{\alpha}[1 + b\omega_{\alpha}(\theta)\theta/|\theta|] \quad \theta \neq \theta \quad (1)$$

where $0 < \alpha \leq 2$, $-1 \leq b \leq 1$, $c \geq 0$ and

$$\omega_{\alpha}(\theta) = \begin{cases} \tan(\pi\alpha/2) & \text{if } \alpha \neq 1 \\ (2/\pi)\log|\theta| & \text{if } \alpha = 1 \end{cases} .$$

α is called the index of the stable law, b is a shape parameter, λ gives the drift, and c is a scaling constant.

<u>Definition.</u> A distribution F is in the domain of attraction of a (nondegenerate) distribution G if there are constants $a_n > 0$, b_n so that $F^{n*}(a_n x + b_n) \Rightarrow G(x)$. (Here F^{n*} is the n-fold convolution of F.)

<u>Theorem 3</u>. The distribution F belongs to the domain of attraction of a normal law $(\alpha = 2)$ if and only if as $n \to \infty$

$$n^2 \int_{|x|>n} F(dx) / \int_{|x|<n} x^2 F(dx) \to 0 \quad .$$

F belongs to the domain of attraction of a stable law of index $0 < \alpha < 2$ if and only if

$$[1-F(x)]/[1-F(x)+F(-x)] \to P \quad \text{as} \quad x \to \infty$$

and

$$1-F(x)+F(-x) \quad = \quad x^{-\alpha} L(x)$$

From the proof of this result in [29], pp. 175-180 we can conclude the scaling constants a_n are of the form $n^{1/\alpha} L(n)$ and satisfy

$$n[1-F(a_n x)+F(-a_n x)] \quad \to \begin{cases} \tilde{c} x^{-\alpha} & \text{if } \alpha < 2 \\ 0 & \text{if } \alpha = 2 \end{cases}$$

The centering constants can be chosen to be

$$
\begin{array}{ll}
nEX_1 & \text{if } 1 < \alpha \le 2 \\
nE(-a_n \vee X \wedge a_n) & \text{if } \alpha = 1 \quad \text{(see [24], p. 315)} \\
0 & \text{if } 0 < \alpha < 1
\end{array}
$$

From Theorem 3 it is immediate that if $S_0 = 0$ and $(S_n - b_n)/a_n \Rightarrow Y$ then the finite dimensional distributions of $V_n(t) = (S_{[nt]} - b_{[nt]})/a_n$ converge. Skorohod has shown (Theorem 2.7 in [32]) that there is also weak convergence.

Theorem 4. If S_n is a random walk and $(S_n - b_n)/a_n \Rightarrow Y$ (nondegenerate) *then* $V_n \Rightarrow V$ *a process with stationary independent increments which has* $V^0(1) \stackrel{d}{=} Y$.

If $\lim_{n\to\infty} b_n/a_n = \mu$ (finite) the centering is unnecessary and S_n/a_n satisfies (i)-(ii).

The next step is to check that (iii) holds. To do this we observe that if $P^y\{T_0 > t\} = 0$ for some positive y then from (8) of Section 3.1, $\{V^x(t), t < T_0\}$ is decreasing. Since V has independent increments this means $\{V^x(t), t \ge 0\}$ is decreasing and so $P\{V^y(t) \le 0\} = 1$.

Conditions for stable processes to have this property are well-known. Using results from [28] we see that if $P^y\{T_0 > t\} = 0$ then $0 < \alpha < 1$, $b = -1$, and $\lambda < 0$ in (1). To complete the proof we will use the scaling relationship to show that none of these processes can occur as limits in (ii).

Let $\phi_t(\theta) = E\exp(i\theta V^0(t))$. Since V^0 has stationary independent increments $\phi_t(\theta) = \phi_1(\theta)^t$. From scaling $V^0(t) \overset{d}{=} t^\gamma V^0(1)$ so $\phi_t(\theta) = \phi_1(t^\gamma\theta)$. Using $t \log \phi_1(\theta) = \log \phi_1(t^\gamma\theta)$ in (1) gives

For limits of S_n/a_n, $\lambda = 0$ if $\alpha \neq 1$ and $b = 0$ if $\alpha = 1$ **(2)**

Since these conditions are incompatible with the ones given above we have shown that (iii) holds.

To prove that (iv) holds we start by observing that stable laws have continuous distributions ([29], p. 183) so $P^x\{T_0 = t\} \le P\{V^x(t) = 0\} = 0$. If $P\{V^0(1) < 0\} > 0$ then the results of Section 3.1 can be applied to give (iv). If $P\{V^0(1) \ge 0\} = 1$ then $P^x\{T_0 > t\} \equiv 1$ for all $x > 0$ and (iv) follows from remarks after Theorem 2.4.

Using (14) of Section 3.1 we see that (v) is satisfied in the first case but not in the second. Having established that (i)-(v) hold when V is not increasing, the next step is to give conditions for the sequence V_n^+ to be tight.

Theorem 5. If X_1 has a distribution F so that $F^{n*}(c_n\cdot) \Rightarrow G$, a stable law with $G(0) < 1$ then V_n^+ is tight for $x_n \equiv 0$.

Remark. If $G(0) = 1$, V is decreasing so $(V^\varepsilon | T_0 > 1) \le \varepsilon$ and $(V^\varepsilon | T_0 > 1) \Rightarrow 0$ as $\varepsilon \downarrow 0$. From the Remark after Theorem 3.8, we see that 0 is the only possible limit in this case.

Proof.

The proof will be given in three lemmas, each of which assumes the hypothesis of Theorem 5 and uses the notation of Theorem 3.9.

Lemma 1. If $G(0) = \beta < 1$ then $Q^0\{N > n\} = n^{-\beta}L(n)$.

Proof.

Since stable laws have continuous distributions $\lim_{k\to\infty} Q^0\{S_k > 0\} = 1-\beta$. By a formula due to Spitzer ([33], p. 330) if S_k is a random walk then

$$\sum_{n=0}^{\infty} Q^0\{N > n\} t^n = \exp\left(\sum_{k=1}^{\infty} \frac{t^k}{k} P\{S_k > 0\}\right)$$

Writing $\theta(t)$ for the generating function of $Q^0\{N > n\}$ and factoring the right hand side gives

$$\theta(t) = (1-t)^{\beta-1}\exp\left(\sum_{k=1}^{\infty} \frac{t^k}{k} (P\{S_k > 0\} - (1-\beta))\right)$$

Now $L(1/1-t) = \exp\left(-\sum_{k=1}^{\infty} \frac{t^k}{k} a_k\right)$ is slowly varying whenever $\lim_{k\to\infty} a_k = 0$ (for a proof see [15], p. 1159) so applying a Tauberian theorem ([24], p. 447) gives

$$\sum_{m=1}^{n} P\{N > m\} = n^{1-\beta}L(n)$$

Since $P\{N > m\}$ is a decreasing function of m, applying a generalization of Landau's theorem ([24], p. 446) gives

$$\lim_{n\to\infty} P\{N > n\} / \frac{1}{n} \sum_{k=1}^{n} P\{N > k\} = 1-\beta$$

so if $\beta < 1$, $P\{N > n\} = n^{-\beta}L(n)$.

Lemma 2. Condition (3a) of Theorem 3.3 is satisfied whenever the limit process has $P\{V^0(1) > 0\} > 0$. If $\alpha = 2$, we have in addition that $(V_n^+(T_\varepsilon) - \varepsilon)^+ \Rightarrow 0$ so tightness follows from Theorem 3.7.

Proof.

Let $X_i = S_i - S_{i-1}$. Let $I_n^y = \inf\{i \le n : X_i/c_n > y\}$, with $I_n^y = \infty$ if the set is empty.

$$P\{N > n, I_n^y < \infty\} \le \sum_{i=1}^{n} P\{N > i-1 \mid I_n^y = i\} P\{I_n^y = i\} \quad .$$

Given $I_n^y = i$, $X_1, \dots X_{i-1}$ are independent and have common distribution function $H_y(x) = (F(x)/F(yc_n)) \wedge 1$. Now $H_y(x) \ge F(x)$ for all x so if $U_1, U_2 \dots U_{i-1}$ are independent random variables each with a uniform distribution on $(0,1)$ then

$$((X_1, \dots X_{i-1}) \mid I_n^y = i) \stackrel{d}{=} (H_y^{-1}(U_1), \dots H_y^{-1}(U_{i-1}))$$

$$\le (F^{-1}(U_1), \dots F^{-1}(U_{i-1})) \stackrel{d}{=} (X_1, \dots, X_{i-1})$$

where the equalities are between distributions and the inequality holds almost surely. From this it is clear that $P\{N > i-1 \mid I_n^y = i\} \le P\{N > i-1\}$. Using this in the first inequality we get

$$P\{I_n^y < \infty \mid N > n\} \le \sum_{i=1}^{n} \frac{P\{N > i-1\}}{P\{N > n\}} P\{I_n^y = i\}$$

Now $P\{N > n\} = n^{-\beta} L(n)$ and $P\{I_n^y = i\} \le P\{X_1 > yc_n\}$ so

$$P\{I_n^y < \infty \mid N > n\} \le \frac{\sum_{i=1}^{n} i^{-\beta} L(n)}{n(n^{-\beta} L(n)} n(1 - F(yc_n))$$

$u(x) = [x+1]^{-\beta}L([x+1])$ is regularly varying with exponent > -1, so from Karamata's theorem

$$\frac{\sum_{i=1}^{n} i^{-\beta}L(n)}{n(n^{-\beta}L(n))} = \frac{\int_0^n u(x)\,dx}{nu(n)} \to 1/1-\beta$$

From Theorem 3 if $0 < \alpha < 2$

$$\lim_{x\to\infty} \frac{1-F(x)}{1-F(x)+F(-x)} = p\in[0,1]$$

and $\lim_{n\to\infty} n[1-F(c_n y)+F(-c_n y)] = cy^{-\alpha}$ so in this case

$\overline{\lim}_{n\to\infty} P\{I_n^y < \infty | N > n\} \leq pcy^{-\alpha}/(1-\beta)$. From this we get

$$\lim_{y\to\infty} \overline{\lim}_{n\to\infty} P\{V_n^+(T_\varepsilon) > y+\varepsilon\} \leq \lim_{y\to\infty} \overline{\lim}_{n\to\infty} P\{I_n^y < \infty | N > n\} = 0$$

so (3a) is satisfied for $0 < \alpha < 2$.

To prove the result for $\alpha = 2$ we observe that from above

$$\overline{\lim}_{n\to\infty} P\{I_n^y < \infty | N > n\} \leq 2\, \overline{\lim}_{n\to\infty} n(1-F(yc_n))$$

so using Theorem 3 gives $(V_n^+(T_\varepsilon) - \varepsilon)^+ \Rightarrow 0$ and applying Theorem 3.7 gives that the sequence is tight when $\alpha = 2$.

To complete the tightness proof when $0 < \alpha < 2$ we use Theorem 3.5 and the following.

Lemma 3. $\lim_{u\downarrow 0} \overline{\lim_{n\to\infty}} P\{V_n^+(u) > y\} = 0.$

Proof.

If $k_n = n-[nst]$ then

$$P\{V_n(st) > y, N > n\} = \int_{(y,\infty)} P\left(\frac{c_{nt}}{c_n} V_{nt}\left(\frac{snt}{[nt]}\right) \in dx, N > snt\right) P(N > k_n \mid v_0 = xc_n)$$

If $m_n = nt - [nst]$ we have

$$P\left(\frac{c_{nt}}{c_n} V_{nt}^+\left(\frac{snt}{[nt]}\right) \in dx\right) = \frac{P\left(\frac{c_{nt}}{c_n} V_{nt}\left(\frac{snt}{[nt]}\right) \in dx, N > snt\right) P(N > m_n \mid v_0 = xc_n)}{P\{N > nt\}}$$

Using the last two equations gives

$$P\{V_n^+(st) > y\} = \frac{P\{N > nt\}}{P\{N > n\}} \int_{(y,\infty)} P\left(\frac{c_{nt}}{c_n} V_{nt}^+\left(\frac{snt}{[nt]}\right) \in dx\right) \frac{P(N > k_n \mid v_0 = xc_n)}{P(N > m_n \mid v_0 = xc_n)}$$

$$\leq \frac{P\{N > nt\}}{P\{N > n\}} P\left\{\frac{c_{nt}}{c_n} V_{nt}^+\left(\frac{snt}{[nt]}\right) > y\right\}$$

From Lemma and Theorem 3.5, V_n^+ is tight in $D[-1,1]$ so for any subsequence there is a further subsequence with $V_{n_k}^+ \Rightarrow V^*$ in $D[-1,1]$. Since for any $s_m \downarrow 0$ we can pick a $t < 1$ with $P\{V^*(s_m t) \neq V^*(s_m t-)$ for some $m \geq 1\} = 0$ the above gives (for appropriate values of y)

$$\lim_{n\downarrow 0} \overline{\lim_{n\to\infty}} P\{V_n^+(u) > y\} \leq t^{-\beta} \lim_{s\downarrow 0} P\{t^{1/\alpha} V^*(st) > y\}$$

$$= t^{-\beta} P\{V^*(0) > yt^{-1/\alpha}\}$$

Since $P\{V^*(0) > z\} \le \overline{\lim}_{n\to\infty} P\{I_n^z < \infty | N > n\}$ using an inequality from the proof of Lemma 2 gives

$$t^{-\beta} P\{V^*(0) > yt^{-1/\alpha}\} \le pcy^{-\alpha} t^{1-\beta}$$

and we can complete the proof by letting $t \downarrow 0$.

At this point we have given conditions for V_n^+ to be tight and $Q^0\{N > n\}$ to be regularly varying so from Theorem 3.10 to prove the conditional limit theorem in the case $G(0) \in (0,1)$ it only remains to show $\lim_{x\downarrow 0} (V^x | T_0 > 1)$ exists.

Theorem 6. If V is a stable process which can occur as a limit in (ii) then $\lim_{x\downarrow 0} (V^x | T_0 > 1)$ exists.

Proof.

If V is decreasing or $P^x\{T_0 > 1\} \equiv 1$ then the result is trivial so for what follows we will assume $P^x\{T_0 > t\} \not\equiv 1$ and hence $P^x\{T_0 > 1\} \downarrow 0$ as $x \downarrow 0$.

Let $R_0^\varepsilon = 0$ and for $k \ge 0$

$$R_{k+1}^\varepsilon = \inf\{t > R_k^\varepsilon : V^0(t+R_k^\varepsilon) - V^0(t) \le -\varepsilon\}$$

Since V^0 has independent increments $R_{k+1}^\varepsilon - R_k^\varepsilon$, $k \ge 0$ are independent and identically distributed. Since $P\{R_1^\varepsilon \le t\} = P^\varepsilon\{T_0 \le t\} \to 1$ as $t \to \infty$ each $R_k^\varepsilon < \infty$ P^0 almost surely.

Let $K_\varepsilon = \inf\{k \ge 1 : R_k^\varepsilon - R_{k-1}^\varepsilon > 1\}$. From (iii), $P\{R_1^\varepsilon > 1\} = P^\varepsilon\{T_0 > 1\} > 0$ so M_ε and $S_\varepsilon = R_{K_\varepsilon}$ are finite P^0 almost surely. Let $U^\varepsilon(t) = \varepsilon + [V^0(S^\varepsilon + t) - V^0(S^\varepsilon)]$. Since V^0 has independent

increments it follows from the construction that $U^{\varepsilon} \stackrel{d}{=} (V^{\varepsilon} | T_0 > 1)$ (see Lemma 2 of Section 4.3 for a detailed proof of a similar result). To show that $(V^{\varepsilon} | T_0 > 1)$ converges weakly as $\varepsilon \downarrow 0$ we will show S^{ε} and U^{ε} converge P^0 almost surely.

Let $m(t) = \inf_{0 \le s \le t} V^0(s)$. Let $S = \inf\{t : m(t) = m(t+1)\}$. Since we have assumed V^0 is not decreasing $P\{V^0(t) = m(t)\} < 1$ and it follows from (iii) that $P^0\{S < \infty\} = 1$.

<u>Lemma 1</u>. $\lim_{\varepsilon \downarrow 0} S^{\varepsilon} \ge S$, P^0 almost surely.

<u>Proof</u>.

Suppose $S^{\varepsilon_m} \to t < \infty$. By choosing a subsequence we can guarantee that either $S^{\varepsilon_m} \ge t$ for all m or $S^{\varepsilon_m} < t$ for all m. If $S^{\varepsilon_m} \downarrow t$, it follows from the right continuity of V^0 and the definition of S^{ε_m} that $m(t) = m(t+1)$ so $S \le t$.

To prove $S \le t$ in the second case observe that if $\delta > 0$ and $t - S^{\varepsilon_m} < \delta$

$$-\varepsilon_m \le \inf_{0 \le s \le 1} V^0(S^{\varepsilon_m} + s) - V^0(S^{\varepsilon_m})$$

$$\le \left[\inf_{0 \le s \le 1-\delta} V^0(t+s)\right] - V^0(S^{\varepsilon_m})$$

$$\to \left[\inf_{0 \le s \le 1-\delta} V^0(t+s)\right] - V^0(t-)$$

so $m(t) = m(t+1-\delta)$ for all $\delta \ge 0$.

To conclude $m(t) = m(t+1)$ it suffices to show $V^0(t+1) = V^0((t+1)-)$. To do this we observe $\max_{1 \le m \le n} (S^{\varepsilon_m} + 1)$ is an increasing

sequence of stopping times which are less than $t+1$ so the desired conclusion follows from the "quasi left continuity" of V (see [22], p. 45 and Exercise I.9.14).

Lemma 2. $\overline{\lim}_{\varepsilon\downarrow 0} S^{\varepsilon} \le S$ P^0 almost surely.

Proof.

Let $X = m(S)$. The first step is to show $X = V^0(S-)$ P^0 almost surely. To do this we observe:

(a) If T is a positive random variable and $\delta > 0$ then there is a stopping time Q_δ so that $P\{Q_\delta \neq T, V^0(T-) > V^0(T)\} \le \delta$ and

(b) if Q is a stopping time and $P^0\{T_0^- = 0\} = 1$ then $P\left\{\inf_{Q\le s\le Q+1} f(s) = f(Q)\right\} = 0$ so $P\{S=Q\} = 0$.

Now R^{ε}_{k+1} is the first time $m(t) - m(R^{\varepsilon}_k) < -\varepsilon$ so we have for all ε there is a K'_ε so that $V^0(R^{\varepsilon}_{K'_\varepsilon}) \in [X-\varepsilon, X]$. Since $K_\varepsilon \le K'_\varepsilon$ this shows $\overline{\lim}_{\varepsilon\downarrow 0} S^{\varepsilon} \le S$ P^0 almost surely.

Having shown $S^{\varepsilon} \to S$, to show $U^{\varepsilon} \to U = V^0(S+t)$, we need to prove $V^0(S) = V^0(S-)$. Although this is obvious the details are tedious to write out so we will refer the reader to Lemma 3.2 of [31] to complete the proof.

Remark. Although this completes the proof of the conditioned limit theorem in the case $G(0) \in (0,1)$, our solution is still somewhat incomplete because we have not given the distribution of the limit. If V is Brownian motion the formulas can be found in [26]. If V is a stable process, however, the distribution of the limit is known only in one special case (see Section 4.5).

4.2 Branching Process

Let $z_n, n \geq 0$ denote the number of particles in the n^{th} generation of a Galton Watson process with $z_0 = 1$ and particle production governed by the probability distribution $\{p_i, i=0,1,2\ldots\}$. (For a detailed definition consult the first few pages of [34] or [35].) Let $f(s) = \sum_{i=0}^{\infty} p_i s^i$ be the generating function of z_1 and for each $n \geq 2$ let $f_n(s) = f(f_{n-1}(s))$ be the generating function of z_n. Kesten, Ney, and Spitzer ([34], p. 19) have shown that

Theorem 1. If $Ez_1 = 1$ and $E(z_1 - 1)^2 = 2\lambda \in (0,\infty)$ then

$$\lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{1-f_n(s)} - \frac{1}{1-s}\right] = \lambda \tag{1}$$

uniformly for $0 \leq s < 1$.

Setting $s = 0$ in (1) and noting that $P\{z_n > 0\} = 1 - f_n(0)$ we obtain the following formula for $P\{z_n > 0\}$.

Theorem 2. As $n \to \infty$ $P\{z_n > 0\} \sim (n\lambda)^{-1}$ **(2)**

Another immediate consequence of Theorem 1 is the following conditioned limit theorem.

Theorem 3. $\lim_{n\to\infty} P\{z_n/n\lambda > x \mid z_n > 0\} = e^{-x}$ **(3)**

Proof.

$$E(e^{-\alpha z_n/n\lambda} \mid z_n > 0)$$

$$= E(e^{-\alpha z_n/n\lambda}; z_n > 0)/E(1; z_n > 0)$$

$$= (f_n(e^{-\alpha/n\lambda}) - f_n(0))/(1 - f_n(0))$$

$$= 1 - (1 - f_n(e^{-\alpha/n\lambda}))/(1 - f_n(0))$$

From (1) $\lim_{n\to\infty} [n(1-f_n(e^{-\alpha/n\lambda}))]^{-1} = \lambda + \lim_{n\to\infty} [n(1-e^{-\alpha/n\lambda})]^{-1}$ and from

(2) $\lim_{n\to\infty} n(1-f_n(0)) = 1/\lambda$ so

$$\lim_{n\to\infty} (e^{-\alpha z_n/n\lambda} \mid z_n > 0) = 1 - (1/(1+\frac{1}{\alpha})) = 1/\alpha+1$$

which completes the proof.

Using the last two results we can compute the limit of $E(e^{-\alpha z_n/n\lambda} \mid z_0 = y_n\lambda n)$. Since the $y_n\lambda n$ ancestors act independently, we have from Theorem 2 that if $y_n \to y \geq 0$ then the number of ancestors which have offspring alive at time n tends to have a Poisson distribution with mean y. Using Theorem 3 now gives that if $y_n \to y \geq 0$

$$\lim_{n\to\infty} E(e^{-\alpha z_n/n\lambda} \mid z_0 = y_n\lambda n) = e^{-y}\sum_{k=0}^{\infty} \frac{y^k}{k!}(1+\alpha)^{-k} = \exp(-y\alpha/1+\alpha)$$

Using the Markov property and Theorem 2.1 it is easy to compute that the finite dimensional distributions of $Z_n^{y_n} = (z_{[n\cdot]}/n\lambda \mid z_0 = y_n\lambda n)$ converge (a result due to Lamperti [36], Theorem 2.5). In [37], Lindvaal has shown that the sequence is tight so we have the following.

<u>Theorem 4.</u> If $y_n \to y \geq 0$ then $Z_n^{y_n} \Rightarrow (Z \mid Z(0) = y)$ where Z is a nonnegative diffusion with transition probabilities satisfying

$$\int e^{-\alpha y} P(Z(t+s) \in dy \mid Z(s) = x) = \exp(-x\alpha/1+\alpha t)$$

for all nonnegative $x, s,$ and t.

Observe that 0 is an absorbing state so

$$P^x\{T_0 > t\} = P^x\{Z(t) > 0\} = 1-e^{-x/t} > 0 \tag{6}$$

and we have that (iii) holds. From the remarks after Theorem 3 $P_n^{x_n}\{N > nt_n\} \to 1-e^{-x/t}$ when $x_n \to x > 0$ and $t_n \to t > 0$ so (iv) and (v) hold.

At this point we have completed our preparation and can apply Theorem 3.9 to conclude:

<u>Theorem 5</u>. $Z_n^+ = (z_{[n\cdot]} \wedge n \mid z_0 = 1, z_n > 0) \Rightarrow (Z^+ \mid Z^+(0) = 0)$ where Z^+ is a Markov process with

$$P(Z^+(t) \in dx) = t^{-2} e^{-x/t}[1-e^{-x/(1-t)}]$$

and

$$P(Z^+(t) \in dy \mid Z^+(s) = x) = x(t-s)^{-2} e^{-(x+y)/(t-s)} \sum_{k=1}^{\infty} \frac{(xy/(t-s)^2)^{k-1}}{k!\,(k-1)!}$$

$$\cdot \frac{1 - e^{-y/1-t}}{1 - e^{-x/1-s}}$$

<u>Proof</u>.

From Theorem 3.9 we have that the finite dimensional distributions of Z_n^+ converge. To obtain the formulas given above from those in Section 3.3 use (2), (3), and (6) of this section and note that from the discussion following Theorem 3

$$P(Z(t+s) \in dy, Z(t+s) > 0 \mid Z(s) = x)$$

$$= \sum_{k=1}^{\infty} e^{-x/t} \frac{(x/t)^k}{k!} \cdot \left(\frac{t^{-1}(y/t)^{k-1}}{(k-1)!} e^{-y/t} \right)$$

To prove that the sequence is tight we have to check that for the distributions given above $Z^+(t) \Rightarrow 0$ as $t \to 0$. To do this we observe that if $y > 0$ and $t \to 0$ then

$$P(Z^+(t) > y) \le \int_y^\infty t^{-2} e^{-x/t} dx = t^{-1} e^{-y/t} \to 0 .$$

4.3 Birth and Death Processes

We will call an integer valued Markov process $\{U(t), t \geq 0\}$ a birth and death process if starting from state j, U remains there for a random length of time having an exponential distribution with mean $(\lambda_j + u_j)^{-1}$ and upon leaving j, U moves to states $j-1$ and $j+1$ with probabilities $u_j(\lambda_j+u_j)^{-1}$ and $\lambda_j(\lambda_j+u_j)^{-1}$ respectively.

It is easy to see that if a birth and death process satisfies (ii) then the limit is a strong Markov process with continuous paths, or a diffusion. In [41], Stone has identified which diffusions can occur as limits in (ii) and given necessary and sufficient conditions for the convergence of birth and death processes to these limits.

As the reader can imagine these conditions are different when the state space of the limit process is $(-\infty,\infty)$ and $[0,\infty)$ and in the latter case also depend upon the nature of the boundary at 0. To keep things simple we will give the results first in the case the state space is $(-\infty,\infty)$ and the diffusion is regular and then consider the other possibilities.

Definition. Let $\tau_x = \inf\{t \geq 0 : V(t) = x\}$. A diffusion V with state space $(-\infty,\infty)$ is regular if $P^x\{\tau_y < \infty\} > 0$ for all x,y.

Theorem 1. ([41], pp. 51-58) A necessary and sufficient condition that there exist a strictly increasing sequence c_n such that as $n \to \infty$ $U(n\cdot)/c_n$ converges (in the sense of (ii)) to a regular diffusion on $(-\infty,\infty)$ is that the sequence defined by $\pi_n = \pi_{n-1}\lambda_{n-1}/u_n$, $\pi_1 = 1$

satisfy $(\lambda_n \pi_n)^{-1} = n^{\alpha_1 - 1} L_1(n)$ and $\pi_n = n^{\alpha_2 - 1} L_2(n)$ where the $\alpha_i > 0$ and the L_i have $\lim_{y\to\infty} L_i(xy)/L_i(y) = 1$ for all $x > 0$ and $\lim_{x\to\infty} L_i(-x)/L_i(x) = d_i \in (0,\infty)$.

In this case $c_n = n^{1/(\alpha_1+\alpha_2)} L(n)$ and the limit process is a diffusion with scale J and speed measure is given by

$$J(x) = \begin{cases} Ax^{\alpha_1} & x \geq 0 \\ -d_1 A|x|^{\alpha_1} & x < 0 \end{cases}$$

$$m(x) = \begin{cases} Bx^{\alpha_2} & x \geq 0 \\ -d_2 B|x|^{\alpha_2} & x < 0 \end{cases}$$

where A and B are positive constants.

Note: To work with this theorem we will have to use some facts about the speed and scale measures of diffusions. A complete discussion of this topic is given in [38], but very little of the information given there is needed to prove our conditioned limit theorems. A readable summary of the results we will need is given in Section 4 of [39].

To show that (iii) holds we observe that if $P^y\{T_0 > t\} > 0$ then from (8) of Section 3.1, $V^y(t \wedge T_0)$ is decreasing for each $t > 0$. Since V has continuous paths and the strong Markov property this implies $P^y\{\tau_z < \infty\} = 0$ for $z > y$, which contradicts the assumed regularity.

To prove (iv) we will use (17) of Section 3.1. Since V is

regular, $V^0 \not\geq 0$ and it follows from the scaling relationship that $P\{V^0(1) > 0\}=0$. To establish that $P^x\{T_0 = t\} = 0$ we recall that Itô and McKean (see Section 4.11 of [38]) have shown that the transition functions of a diffusion have densities with respect to the speed measure so

$$P^x\{T_0 = t\} \leq P\{V^x(t) = 0\} \leq m(\{0\}) = 0$$

Since V is regular $P^x\{T_0 > t\} \not\equiv 1$ and from (13) it follows that $\lim_{x\downarrow 0} P^x\{T_0 > t\} = 0$ for all $t > 0$. Since $P\{N > m | v_0 = x\}$ is an increasing function of x and (iv) holds using (14) gives that (v) holds.

Having established (i)-(v) we will now prove the conditioned limit theorem by checking the hypotheses of Theorem 3.10. The first two steps are easy. Since $(V_n^+(T_\varepsilon) - \varepsilon)^+ \leq 1/c_n \to 0$ it is immediate from Theorem 3.7 that V_n^+ is tight for $x_n \to 0$. To get the asymptotic formula for $Q^0\{N > n\}$ we observe that from [40] p. 253 we have $Q^0\{N > n\} = n^{-\beta}L(n)$ where $\beta = \alpha_1/\alpha_1+\alpha_2$.

To complete the proof we have to show:

Theorem 2. If V is a diffusion which can occur as a limit in Theorem 1 then $\lim_{x\downarrow 0} (V^x | T_0 > 1)$ exists.

Proof.

Suppose V is defined on a probability space with σ-fields $\mathcal{F}_t = \sigma\{V(s):s \leq t\}$ and shift operators $\{\theta_t; t \geq 0\}$. Let $S_\varepsilon = \inf\{s > 0: V(s) = \varepsilon, V(u) > 0 \text{ for } s < u \leq s+1\}$ and let $Z_\varepsilon(t) = V(S_\varepsilon+t)$.

Lemma 1. For $\varepsilon > 0$ and all x $S^{\varepsilon} < \infty$ P^x almost surely. As $\varepsilon \downarrow 0$ $S_{\varepsilon} \downarrow S_0$ and $Z_{\varepsilon} \to Z_0$ P^0 almost surely.

Proof.

For $\varepsilon > 0$ let $R_{\varepsilon}^0 = -1$ and $R_{\varepsilon}^{k+1} = \inf\{t \geq R_{\varepsilon}^k + 1 : V(t) = \varepsilon\}$. If $y \neq \varepsilon$ then from [39], p. 53:

$$P^y\{R_{\varepsilon}^1 < \infty\} = \lim_{M\to\infty} P^y\{\tau_{\varepsilon} < \tau_{(y-\varepsilon)M}\}$$

$$= \lim_{M\to\infty} \frac{J(x) - J((y-\varepsilon)M)}{J(\varepsilon) - J((y-\varepsilon)M)} = 1$$

so using the strong Markov property and induction gives that $P^x\{R_{\varepsilon}^k < \infty\} = 1$ for all x and k. Now if V has no zero in $[R_{\varepsilon}^k, R_{\varepsilon}^k + 1]$ then $S_{\varepsilon} \leq R_{\varepsilon}^k$ so

$$P^x\{S_{\varepsilon} \leq R_{\varepsilon}^k | S_{\varepsilon} > R_{\varepsilon}^{k-1}\} \geq P^{\varepsilon}\{T_0 > 1\} > 0$$

and hence

$$P^x\{S_{\varepsilon} < \infty\} = 1 .$$

For $0 \leq \delta < \varepsilon$, $S_{\delta} \leq \sup\{t < S_{\varepsilon} : V(t) = \delta\}$ so $S_{\varepsilon}\downarrow$ as $\varepsilon\downarrow$. To see that $S_{\varepsilon} \downarrow S_0$ note that

$$\rho = \inf\{t - S_0 - 1 : t > S_0, V(t) = 0\} > 0$$

so $S_{V(S_0+\lambda\rho)} - S_0 \leq \lambda\rho$ for all $0 < \lambda < 1$. Since V has continuous paths and $Z_{\varepsilon}(t) = V(S_{\varepsilon}+t)$, $S_{\varepsilon}\downarrow 0$ implies $Z_{\varepsilon} \to Z_0$.

Having proven Lemma 1 to complete the proof of Theorem 2 it suffices to show:

Lemma 2. For $\varepsilon > 0$ Z_ε and $(V^\varepsilon | T_0 > 1)$ have the same distribution.

Proof.

Let F be a Borel subset of D. Clearly,

$$P\{Z_\varepsilon \in F\} = P\{Z_\varepsilon \in F, S_\varepsilon = \tau_\varepsilon\} + P\{Z_\varepsilon \in F, S_\varepsilon > \tau_\varepsilon\} \tag{1}^*$$

Since τ is a stopping time and V a strong Markov process

$$\begin{aligned} P\{Z_\varepsilon \in F\} &= E[P(Z_\varepsilon \in F, S_\varepsilon = \tau_\varepsilon | \mathcal{F}_{\tau_\varepsilon})] \\ &= P\{V^\varepsilon \in F, T_0 > 1\} \end{aligned} \tag{2}$$

If $S_\varepsilon > \tau_\varepsilon$ then $V(s) = 0$ for some $s \in (\tau_\varepsilon, \tau_\varepsilon + 1]$. Letting $\tau'_\varepsilon = \inf\{s : s \in (\tau_\varepsilon, \tau_\varepsilon + 1], V(s) = 0\}$ where $\tau'_\varepsilon = \infty$ if the last set is empty, we have

$$\begin{aligned} P\{Z_\varepsilon \in F, S_\varepsilon > \tau_\varepsilon\} &= P\{Z_\varepsilon \in F, \tau'_\varepsilon < \infty\} \\ &= E[\tau'_\varepsilon < \infty; E(1_{\{Z_\varepsilon \in F\}} | \mathcal{F}_{\tau'_\varepsilon})] \end{aligned} \tag{3}$$

On the set $\{\tau'_\varepsilon < \infty\}$, $1_{\{Z_\varepsilon \in F\}}$ can be written as $\phi(\theta_{\tau'_\varepsilon})$ so from (3) and the strong Markov property we get

$$\begin{aligned} P\{Z_\varepsilon \in F, S_\varepsilon > \tau_\varepsilon\} &= (E^0\phi)P\{\tau'_\varepsilon < \infty\} \\ &= P\{Z_\varepsilon \in F\}(1 - P^\varepsilon\{T_0 > 1\}) \end{aligned} \tag{4}$$

Combining (1), (2), and (4) gives

*Note: when P is written without a superscript the indicated probability is independent of the initial distribution.

$$P\{Z_\varepsilon \in F\} = P\{V^\varepsilon \in F, T_0 > 1\} + P\{Z_\varepsilon \in F\}(1 - P^\varepsilon\{T_0 > 1\})$$

so

$$P\{Z_\varepsilon \in F\} = P(V^\varepsilon \in F | T_0 > 1)$$

which proves Lemma 2.

This completes our development for the "regular" case. The next step is to determine in what other cases we can get a nontrivial conditioned limit theorem.

To do this we observe that from (16) and (18) of Section 3.1 either $P^0\{T_0^- = 0\} = 1$ or $V^0 \geq 0$ so if V is not regular there is no loss of generality in assuming the state space is $[0,\infty)$. In Section 3.1 we argued that if 0 was inaccessible from positive levels then the limit theorem is trivial so we will assume $P^x\{T_0 > t\} \not\equiv 1$. In this case (13) of 3.1 implies $\lim_{x \downarrow 0} P^x\{T_0 > t\} = 0$ so (12) of 3.1 gives $P^0\{T_0 = 0\} = 1$. Since $P^0\{T_0^+ = 0\} = 1$ if and only if $P^0\{V^0(1) > 0\} > 0$ there are only boundary possibilities to consider

(a) reflecting: $P^0\{T_0^+ = 0\} = P^0\{T_0 = 0\} = 1$

(b) absorbing: $V^0 \equiv 0$

Conditions for convergence in these cases can be obtained from [41]:

Theorem 3. Let $\{U(t), t \geq 0\}$ be a birth and death process with state space $\{0,1,2...\}$.

If 0 is a reflecting boundary for V then $U(n\cdot)/c_n \Rightarrow V$ and

(iv) holds if and only if the sequence π_n defined in Theorem 1 has $(\lambda_n \pi_n)^{-1} = n^{\alpha_1 - 1} L_1(n)$ and $\pi_n = n^{\alpha_2 - 1} L(n)$ where α_1 and $\alpha_1 + \alpha_2$ are positive and the L_i have $\lim_{\theta\to\infty} L_i(\theta x)/L_i(\theta) = 1$ for all $x > 0$.

If 0 is an absorbing boundary for V and $\lambda_0 = 0$ in U then $U(n\cdot)/c_n \Rightarrow V$ and (iv) holds if and only if in addition to the conditions stated above we have

$$\lim_{x\to 0} \overline{\lim_{n\to\infty}} \int_0^x \frac{u(zc_n)}{u(c_n)} v_n(dz) = 0$$

where $v_n(x) = (v(xc_n) - v(c_n))/(v(2c_n) - v(c_n))$, $v(i) = \sum_{j=1}^{i} \pi_i$ and $u(i) = \sum_{j=1}^{i} (\lambda_j \pi_j)^{-1}$.

In each case $c_n = n^{1/\alpha_1 + \alpha_2} L(n)$ and there are positive constants A and B so that the limit process is a diffusion with scale $J(x) = Ax^{\alpha_1}$ $x \geq 0$ and a speed measure m concentrated on $(0,\infty)$ given by

$$m(x) = \begin{cases} Bx^{\alpha_2} & \text{if } \alpha_2 \neq 0 \\ B \log x & \text{if } \alpha_2 = 0 \end{cases}$$

If $\alpha_2 > 0$, 0 is a reflecting boundary. In the other cases 0 is absorbing.

Since Theorem 3 gives conditions for (ii) and (iv) to hold and the arguments given above for (iii) and (v) still apply, we have that (i)-(v) hold. From Theorem 3.7, V_n^+ is tight for $x_n \to 0$.

If 0 is a reflecting boundary it is easy to use Theorem 3.10 to

show V_n^+ converges: a similar argument works to show $\lim_{\varepsilon \downarrow 0}(V^{\varepsilon} | T_0 > 1)$ exists (we only have to change the proof that $P^x\{S_\varepsilon < \infty\} = 1$) and it follows from [40], p. 253 that $Q^0\{N > n\} = n^{-\gamma_1/(\gamma_1+\gamma_2)} L(n)$.

If 0 is an absorbing boundary, however, both of these arguments fail. We leave it to the interested reader to decide whether the conditioned limit theorem will hold in general in this case.

4.4 The M/G/1 Queue

In the M/G/1 queue customers arrive at the jump times of a Poisson process $A(t)$, $t \geq 0$ with rate λ and have service times which are independent positive random variables with the same distribution.

If ξ_i denotes the amount of service required by the i^{th} customer to arrive after time 0 then $S(t) = \sum_{i=1}^{A(t)} \xi_i$ is the amount of work that has arrived at the facility at time t. If the initial backlog of work is x and the server is not idle at any moment before t then $L(t) = x+S(t)-t$ is the amount of work not completed at time t. If the server has been idle then we have to add to this number the amount of time he has been idle so the amount of work that remains in general is given by $V(t) = L(t) - \left(\min_{0 \le s \le t} L(s)\right)^{+}$.

It is easy to use Donsker's theorem to obtain conditions for U to satisfy (ii).

<u>Theorem 1.</u> Suppose $\lambda E\xi_1 = 1$ and $E(\xi_1 - 1/\lambda)^2 = \sigma^2 \in (0,\infty)$. If $x_n \to x \geq 0$ then $(V(n\cdot)/\sigma n^{1/2} \mid V(0) = x_n \sigma n^{1/2})$ converges to $(\overline{B} \mid \overline{B}(0) = x)$ where $\overline{B}$ is reflecting Brownian motion.

<u>Proof.</u>

$S(t)$ is the sum of a Poisson number of independent random variables with mean $E\xi_1$ so from [20] Theorem 17.2 $(x(n\cdot) - \lambda(E\xi_1)nt)/\sigma n^{1/2}$ converges to a Brownian motion B. From this it follows that if $x_n \to x \geq 0$ $(L(n\cdot)/\sigma n^{1/2} \mid L(0) = x_n \sigma n^{1/2})$ converges to $(B \mid B(0) = x)$

and the desired conclusion now follows from the continuous mapping theorem.

Since the limit in Theorem 1 is reflecting Brownian motion (iii) holds. To see that (iv) and (v) are satisfied we observe that if $x_n \to x \geq 0$ and $t_n \to t > 0$

$$
\begin{aligned}
& P\left(\inf_{0\leq s\leq t_n} V(ns) > 0 \;\middle|\; V(0) = x_n \sigma n^{1/2}\right) \\
= \; & P\left(\inf_{0\leq s\leq t_n} L(ns) > 0 \;\middle|\; L(0) = x_n \sigma n^{1/2}\right) \\
\to \; & P\left(\inf_{0\leq s\leq t} B(s) > 0 \;\middle|\; B(0) = x\right) \\
= \; & P\left(\inf_{0\leq s\leq t} \bar{B}(s) > 0 \;\middle|\; \bar{B}(0) = x\right)
\end{aligned}
$$

Having verified (i)-(v) the next step is to compute the asymptotic formula for the probability of the conditioning event. To do this we will use the Laplace transform.

Let $T_0 = \inf\{t>0 : U(t) = 0\}$

Let $\phi_x(\alpha) = E(e^{-\alpha T_0} \mid U(0) = x)$

Since the arrivals form a Poisson process we have

$$\phi_{x+y}(\alpha) = \phi_x(\alpha)\phi_y(\alpha) \tag{1}$$

and

$$\phi_x(\alpha) = e^{-\alpha x} \int_{(0,\infty)} \phi_y(\alpha) P(S(x) \in dy) \tag{2}$$

From (1) it follows that there is a number $\eta(\alpha)$ so that

$\phi_x(\alpha) = e^{-x\eta(\alpha)}$. Using this fact in (2) gives

$$e^{-x\eta(\alpha)} = e^{-\alpha x}E(e^{-\eta(\alpha)S(x)}) \tag{3}$$

Now if $\theta(\beta) = E(e^{-\beta\xi_1})$ then $E(e^{-\beta S(x)}) = e^{-\lambda x(1-\theta(\beta))}$ so (3) may be written as

$$-x\eta(\alpha) = -\alpha x-\lambda x(1-\theta(\eta(\alpha)))$$

or

$$\eta(\alpha) = \alpha + \lambda-\lambda\theta(\eta(\alpha)) \tag{4}$$

If H is the distribution of ξ_1, Takács ([46], pp. 47-49) has shown that equation (4) has a unique positive solution given by

$$\eta(\alpha) = \alpha + \lambda\left[1- \sum_{j=1}^{\infty} \frac{\lambda^{j-1}}{j!} e^{-(\lambda+\alpha)x}x^{j-1}H^{j*}(dx)\right] \tag{5}$$

where H^{j*} denotes the j-fold convolution of H.

Writing $\gamma(\alpha)$ for the sum in (5) we have

$$\phi_x(\alpha) = e^{-x\eta(\alpha)} = e^{-x(\alpha+\lambda(1-\gamma(\alpha)))}$$

Brody ([43], p. 78) has shown that if $E\xi_1^2 = \mu_2 < \infty$ then

$$1 - \gamma(\alpha) \sim (2/\mu_2)^{1/2}\alpha^{1/2} \quad \text{as} \quad \alpha\downarrow 0$$

so

$$1 - \phi_x(\alpha) \sim x\lambda(2/\mu_2)^{1/2}\alpha^{1/2} \quad \text{as} \quad \alpha\downarrow 0$$

Using a result of Dynkin ([44], p. 179) now shows that

$$P(T_0 > t \mid L(0) = x) \sim x\lambda(2/\pi\mu_2)^{1/2}t^{-1/2} \quad \text{as} \quad x\uparrow\infty$$

At this point we are ready to use Theorem 3.10 to prove the conditioned limit theorem. From results in 4.1 or 4.3 we have that $\lim_{x \downarrow 0} (V^x | T_0 > 1)$ exists so it remains to show that the sequence V_n^+ is tight. To do this we will imitate the proof given in Section 4.1.

Let $J_h^n = \inf\{j \geq 1 : \xi_j > h\sigma n^{1/2}\}$.

$$Q^a\{T_0 > n, J_h^n < \infty\} \leq \sum_{k=1}^{\infty} \int_0^1 P\left(T_0 > ns, J_h^n = A^{-1}(k), \frac{A^{-1}(k)}{n} \in ds\right)$$

$$= \sum_{k=1}^{\infty} \int_0^1 P(T_0 > ns | A^{-1}(k) = ns, J_h^n = A^{-1}(k))$$

$$\times P(J_h^n = A^{-1}(k) | A^{-1}(k) = ns) P\left(\frac{A^{-1}(k)}{n} \in ds\right)$$

$$\leq \sum_{k=1}^{\infty} \int_0^1 P(T_0 > ns | A^{-1}(k) = ns)$$

$$\times P\{\xi_1 \leq h\sigma n^{1/2}\}^{k-1} P\{\xi_1 > h\sigma n^{1/2}\} \frac{(ns)^{k-1} e^{-ns/\lambda^{k-1}}}{k-1!\ \lambda^k} ds$$

Since $P(T_0 > ns | A^{-1}(k) = ns) \leq P(T_0 > ns | A(ns) = k-1)$ the expression above is

$$\leq \lambda^{-1} P\{\xi_1 > h\sigma n^{1/2}\} \sum_{k=1}^{\infty} \int_0^1 P(T_0 > ns | A(ns) = k-1) e^{-ns/\lambda} \frac{(ns/\lambda)}{k-1!} ds$$

$$= \lambda^{-1} P\{\xi_1 > h\sigma n^{1/2}\} \int_0^1 P\{T_0 > ns\} ds$$

Dividing by $Q^a\{T_0 > n\}$ gives

$$Q^a(J_h^n < \infty | T_0 > n) = \lambda^{-1} n P\{\xi_1 > h\sigma n^{1/2}\} \frac{\int_0^1 Q^a\{T_0 > ns\} ds}{n Q^a\{T_0 > n\}}$$

Since ξ_1 has finite variance $nP\{\xi_1 > h\sigma n^{1/2}\} \to 0$ as $n \to \infty$. Now $Q^a\{T_0 > n\} = n^{-1/2}L(n)$ so using Karamata's theorem gives

$$\frac{\int_0^1 Q^a\{T_0 > ns\}\,ds}{nQ^a\{T_0 > n\}} \to 2$$

and $Q^a\{J_h^n < \infty \mid T_0 > h\} \to 0$. To complete the tightness proof we use the same arguments which were used in Section 4.1 for the case $\alpha = 2$.

4.5 Conditioning on $T_B > n$ when B is a Bounded Set

In this section we will extend the results of Chapter 3 to study the effect of conditioning a lattice random walk $S_n = S_0 + \sum_{i=1}^{n} X_i$ on $N_B > n$ where $N_B = \inf\{m \geq 1 : S_m \in B\}$ and B is a finite set. We will leave it to the reader to check that the arguments given below apply to the random walks studied in [14], and that many of the results below hold in the generality suggested by the title.

The organization of this section is the same as that of the previous four. We will first verify that (i)-(v) hold and then show the results of Section 3.3 can be applied to conclude the desired conditioned limit theorems.

Necessary and sufficient conditions for (ii) to hold were given in Section 4.1. From results there

$$P^x\left\{\inf_{0\leq s\leq t} |f(s)| > 0\right\} \geq P^x\{T_0 > t\} > 0$$

so (iii) holds. To check that $(V_n^{x_n} | N_B > nt_n)$ converges when $x_n \to x > 0$ and $t_n \to t > 0$ requires more work.

Let $T_{\{0\}}(f) = \inf\left\{t \geq 0 : \inf_{0\leq s\leq t} |f(s)| = 0\right\}$. Now,

$$\partial\{T_{\{0\}} > t\} = \{T_{\{0\}} = t\} \cup \left\{f : \inf_{0\leq s\leq t} f(s) = 0 \quad \text{or} \quad \sup_{0\leq s\leq t} f(s) = 0\right\}$$

and

$$P^x\{T_{\{0\}} = t\} \leq P\{V^x(t) = 0\} = 0,$$

so if $x > 0$ and $P^0\{T_0^- = 0\} = 1$, it follows from the strong Markov property that $P^x(\partial\{T_{\{0\}} > t\}) = 0$ and $P_n^{x_n}(N_B > n) \to P^x\ \{T_{\{0\}} > t\}$

whenever $x_n \to x > 0$ and $t_n \to t > 0$.

Let $A_n = \{N_B > nt_n\}$. For all $\varepsilon > 0$

$$\text{LIMINF } A_n \supset \left\{f: \inf_{0\le s\le t} |f(s)| > \varepsilon\right\}$$

so

$$\text{LIMINF } A_n \supset \{f: T_{\{0\}}(f) > t\}$$

and using Theorem 2.4 gives $(V_n^{x_n} | N_B > nt_n) \Rightarrow (V^x | T_{\{0\}} > t)$

To show that (v) holds let $T_\varepsilon^c = \inf\{t: V(t) \notin [-\varepsilon,\varepsilon]\}$ and observe that if $x_n \to 0$ and n is sufficiently large

$$P_n^{x_n}\{N_B > n\} \le P_n^{x_n}\{T_\varepsilon^c \ge 1/2\} + E\left[T_\varepsilon < 1/2; P^{V_n^{x_n}(T_\varepsilon^c)}\{N_B > n(1-T_\varepsilon^c)\}\right]$$

If ε is such that T_ε^c is P^0 a.s. continuous

$$\overline{\lim_{n\to\infty}}\, P_n^{x_n}\{N_B > n\} \le P^0\{T_\varepsilon^c \ge 1/2\} + E\left[T_\varepsilon^c < 1/2; P^{V^0(T_\varepsilon^c)}\{T_{\{0\}} > 1-T_\varepsilon^c\}\right]$$

Letting $\varepsilon \downarrow 0$, $P^0\{T_\varepsilon^c \ge 1/2\} \downarrow 0$ and $V^0(T_\varepsilon^c) 1_{\{T_\varepsilon \le 1/2\}} \Rightarrow 0$ so the result follows from the fact that $P^x\{T_{\{0\}} > t\} \downarrow 0$ for all $t > 0$.

If $Q^a\{N_B > n\}$ decreases to a positive limit then the methods of Section 3.3 can be used to show

$$(V_{[n\cdot]}/c_n | V_0 = a, N_B > n) \Rightarrow V^0$$

so for the rest of the section we will assume $Q^a\{N_B > n\} \downarrow 0$, that is, S_n is recurrent.

Using Theorem 3 of Section 4.1 and Theorem 1.3 of [4], a random walk in the domain of attraction of a stable law of index α is recurrent if $\alpha > 1$ and $EX_1 = 0$ or $\alpha = 1$ and $\int_1^\infty P(|X_1| > u)^{-1} u^{-2} du = \infty$, so we will restrict our attention to these cases.

If $\alpha = 1$ then $P^x\{T_{\{0\}} > t\} \equiv 1$ for all $x \neq 0$ (see [28], Theorems 3.1 and 5.4) so from the arguments in Section 3.1 if the limit exists in the sense of (a), $V_n^+ \Rightarrow V^0$. In previous sections we have eliminated such cases but in this instance we will not because the situation has been studied by Belkin and his results indicate there are technical complications which make the "trivial" case the hardest of all.

To describe Belkin's result we have to introduce some of his notation:

for $n \geq 0$ let $Q_B^n(x,y) = P(S_n = y, N_B \geq n | S_0 = x)$

$$\text{let } g_B(x,y) = \sum_{n=0}^{\infty} Q_B^n(x,y) .$$

$g_B(x,y)$ is the expected number of visits to y starting at x up to and including the first visit to B. It is known (see [4]) that

<u>Lemma 1</u>. If $EX_1^2 = \infty$ then

$$g_B(x) = \lim_{|y| \to \infty} g_B(x,y) \text{ exists.}$$

If $EX_1^2 < \infty$ then

$$g_B(x) = \frac{1}{2} \lim_{y \to \infty} [g_B(x,y) + g_B(x,-y)] \text{ exists.}$$

The reason for our interest in this quantity is explained by the next result (which comes from [4], p. 148).

Lemma 2. Let B be a finite set, $B_+ = \{n : n > \sup B\}$, $B_- = \{n : n < \inf B\}$, and $C = B_+ \cup B_-$.

If $EX_1^2 < \infty$ then $g_B(x) > 0$ if and only if $P^x\{N_C < N_B\} > 0$.

If $EX_1^2 = \infty$ then $g_B(x) > 0$ if and only if $P^x\{N_{B_+} < N_B\} > 0$ and $P^x\{N_{B_-} < N_B\} > 0$.

It is clear that $P^x\{N_C < N_B\} > 0$ is necessary for a nondegenerate conditioned limit theorem. The reason for requiring $P^x\{N_{B_+} < N_B\}$ and $P^x\{N_{B_-} < N_B\} > 0$ is less obvious but the need for this condition will be indicated later. To justify assuming $g_B(x) > 0$ at this point, we observe that this condition holds if the limit distribution has $\alpha < 2$ and $|b| < 1$.

Theorem 1. ([4], p. 158) Let B be a finite set with $g_B(0) > 0$ and let F be the distribution of X_1.

If F belongs to the domain of attraction of a stable law of index $1 < \alpha \le 2$ and $EX_1 = 0$, or F belongs to the domain of normal attraction of a stable law of index $\alpha = 1$ and $\lim_{x\to\infty} \int_{-x}^{x} yF(dy) = \mu$ (finite) then for every real number y

$$\lim_{n\to\infty} P(S_n/c_n \le y \mid S_0 = 0, N_B > n) = H_{\alpha,B}(y)$$

where $H_{\alpha,B}$ is a probability distribution with characteristic function

$\psi_{\alpha,B}$ and a density $h_{\alpha,B}$ given by the following formulas:

If $1 \le \alpha < 2$ and $EX_1^2 = \infty$ then

$$\psi_{\alpha,B}(t) = 1 - c|t|^{\alpha}(1+b\omega_{\alpha}(t)) \int_0^1 x^{(1/\alpha)-1}\eta(t(1-x)^{1/\alpha})dx$$

where η is the characteristic function of the limit of $F^{n*}(c_n \cdot)$.

If $\alpha = 2$ and $EX_1^2 = \sigma^2 < \infty$ then

$$h_{2,B}(x) = (2\sigma^2)^{-1}\exp(-x^2/2\sigma^2)[|x| - (xES_{N_B}/\sigma^2 g_B(0))]$$

At this point if we were conditioning on $T_0 > n$ we could use Theorem 3.9 to conclude that

$$V_n^{\#} = (S_{[n\cdot]}/c_n | S_0 = 0, N_B > n)$$

converges weakly to a limit process with finite dimensional distributions given by (2) and (3) of Section 3.3.

Although Theorem 3.9 cannot be applied the same proof can be used to give the result desired. Returning to Section 3.3 we see that to conclude convergence of finite dimensional distributions we needed (in addition to the convergence of $V_n^{+}(1)$ to $v^* \neq 0$) that equation (4) was valid and (iii)-(v) hold; while to prove tightness we needed Theorems 3.3-3.5 of Section 3.2 which hold as long as the post-T_{ϵ} process converges and $\lim_{x\to\infty} P^x\{T_{\{0\}} \le 1\} = 0$.

With slightly more effort we could also prove the convergence of $V_n^{\#}$ using an analogue of Theorem 3.10. Millar ([30], Lemma 4.5) has shown:

Theorem 2. Let p_t be the density of $V^0(t)$. Let f be a bounded Borel function.

Let $Q_t f = \int f(y) q_t(y) dy$ where

$$q_t(y) = (y/t)e^{-y^2/2t} \qquad \text{if } \alpha = 2 \text{ and}$$

$$= p_t(y) + \frac{\alpha t^{1-1/\alpha}}{2\alpha-1} \int_0^t [p_t(y) - p_{t-s}(y)] s^{(1/\alpha)-2} ds \qquad \text{if } 1 < \alpha < 2$$

If $1 < \alpha < 2$ and $|b| < 1$ then

$$\lim_{x \to 0} E^x f(V(t)) 1_{\{T_{\{0\}} > t\}} / P^x\{T_{\{0\}} > t\} = Q_t f$$

If $\beta = 1$ and $x \downarrow 0$ or $\beta = -1$ and $x \uparrow 0$ then the same result holds.

If $\alpha = 2$

$$\lim_{x \to 0} E^x | f(V(t)) | 1_{\{T_{\{0\}} > t\}} / P^x\{T_{\{0\}} > t\} = Q_t f .$$

From this we see that if $1 < \alpha < 2$ and $|b| < 1$ then $\lim_{x \to 0}(V^x | T_{\{0\}} > 1)$ exists and the methods of Theorem 3.10 apply. If $\alpha = 2$ or $|b| = 1$, however, the limits are different for $x \downarrow 0$ or $x \uparrow 0$ (see [30] Lemma 4.6 for the case $|b| = 1$) so to prove convergence with Theorem 3.10 we would have to show $\lim_{t \downarrow 0} \lim_{n \to \infty} P\{V_n^{\#}(t) > 0\}$ exists.

If $\alpha = 2$, we can use Belkin's result. Comparing the limits in

Theorems 1 and 2 in this case we see that for all $t > 0$

$$\lim_{n\to\infty} P\{V_n^{\#}(t) > 0\} = \frac{1}{2}\left(1-(ES_{N_B}/\sigma^2 g_B(0))\right) = \bar{p}_B$$

so $V_n^{\#}$ converges to a process which is positive at all $t > 0$ with probability $\bar{p}_B$ and negative at all $t > 0$ with probability $1-\bar{p}_B$.

If $1 < \alpha < 2$, $|b| = 1$ and $g_B(0) > 0$ a similar analysis can be performed to identify the limit. Since this involves manipulating the transform in Theorem 1 and does not give us much new information, we have not done these calculations.

We close this section by giving an example (due to Belkin [4], pp. 162-163) which shows what can happen when $|b| = 1$ and $g_B(0) = 0$.

<u>Example</u>. Let F be a distribution function for an integer valued random variable X_1 with $P\{X_1 \le -2\} = 0$ and suppose F is in the domain of normal attraction of a stable law of index α with $1 < \alpha < 2$.

Let $B = \{-1\}$. Since $P^0\{N_{B_-} < N_B\} = 0$ it follows from Lemma 2 that $g_B(0) = 0$ and Theorem 1 cannot be applied. There is a good reason for this: the conclusion of Theorem 1 is false. Belkin has shown that $(S_n/c_n | S_n = 0, N_B > n)$ converges to a random variable with characteristic function

$$1 - c|t|^{\alpha}(1+w_{\alpha}(t)) \int_0^1 x^{-(1/\alpha)} \eta(t(1-x)^{1/\alpha})\,dt \quad .$$

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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ABSTRACT

Let U_k be a discrete time Markov process with state space E and let S be a proper subset of E. In several applications it is of interest to know the behavior of the system after a large number of steps given that the process has not entered S. For example if U_k is a branching process a limit theorem of this type gives information about the size of the kth generation given that extinction has not occured by time k.

Seneta and Vere-Jones have given sufficient conditions for the conditioned sequence to converge (without normalization) when the state space is discrete. Their results can be applied to sub-critical branching processes. If the chain is null recurrent or transient, however, all their limits are zero so we have to divide by constants which tend to infinity to obtain an interesting limit theorem.

In this instance the most desirable type of result is a functional limit theorem, i.e. a result asserting the convergence of a sequence of stochastic processes derived from the sequence of observations. This was the goal in several previous studies but in most cases the results obtained are incomplete due to problems with the tightness argument.

It was the presence of these technical difficulties which motivated this investigation. The remedies we have developed allow us to state general conditions for the conditioned processes to converge when S is a half-line or bounded set.

Our results can be applied to null recurrent Galton-Watson branching processes (when the offspring distribution has a finite second moment), to random walks in the domain of attraction of a stable law, to the waiting time process of the M/G/1 queue (when the service distribution has finite second moment), and to birth and death processes in the domain of attraction of a diffusion which is regular or has zero as a reflecting boundary. The limit theorems we obtain in this way generalize and complete several results in the literature.

An important aspect of our methods is that the main theorem is derived from a set of basic assumptions so if a person is interested in a conditioned limit theorem not included in the list above he can apply our results directly instead of adapting one of our proofs to meet his needs.

A second feature of our solution which deserves mention is that in the development of the main theorem we prove a result which gives conditions for the convergence of the conditioned measures $P_n(\cdot|A_n)$ when P_n converges to P and inf $P_n(A_n) > 0$. As the reader may expect our conditions are that the sets A_n converge to A in an appropriate sense and that A may be approximated from the inside (or outside) by sets G_m which are P-continuity sets.